

INTRODUCTION TO ROCKET PROPULSION

L2: Rocket Performance

How much propellant do we need to send a rocket to space?

In this lecture, we'll review the fundamentals of rocket performance. Using Newton's Second Law, we'll determine the thrust of a rocket that continuously expels exhaust. Then we'll learn about specific impulse, which is basically the "fuel efficiency" of a rocket. Finally, we'll derive the Ideal Rocket Equation, which allows us to calculate the propellant mass required to send things to space!

LEARNING GOALS:

1. Quantify the thrust of a rocket that continuously expels mass using Newton's Second Law.
2. Define exhaust velocity, impulse, and specific impulse.
3. Calculate the mass of propellant required to change the velocity of a rocket-propelled vehicle by a specified amount using the Ideal Rocket Equation.

ROCKET THRUST

In this lecture, we will calculate the thrust of a rocket that produces a constant stream of exhaust. In chemical rocket engines, the exhaust is a stream of gas produced by a combustion reaction. This situation needs to be approached differently from the case in which a rocket expels discrete amounts of mass. In the tennis ball problem from last lecture, one chunk of mass was expelled at a particular instant in time. In the steady exhaust stream problem, mass is continuously leaving the rocket at a given rate over a period of time. To address the problem of a constant exhaust stream, we need to define a quantity called the *mass flow rate*.

DEFINITION 2.1 The rate at which mass is exhausted from a rocket is called the **mass flow rate**. It is denoted by the symbol \dot{m} and typically has units of kg/s.

Let's draw a diagram of the problem! Figure 2 shows our rocket flying through space at a velocity, v , and with a mass, m . Both the velocity and mass are functions of time. We expect the mass of the



Figure 1: A launch vehicle with chemical rockets which produce a steady stream of exhaust.

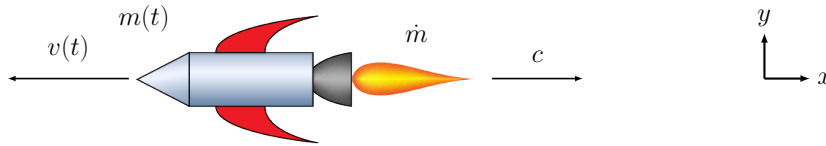


Figure 2: A rocket-propelled vehicle flying through space. The rocket engine continuously expels exhaust at a given mass flow rate with a given exhaust velocity.

vehicle to decrease over time as propellant is expelled from the rocket engine. We also expect the velocity of the vehicle to increase over time as the thrust produced by the rocket engine accelerates it.

We will assume that the propellant is expelled from the vehicle at a given mass flow rate, denoted by \dot{m} . We'll also assume that the propellant is exhausted from the vehicle at a constant velocity, denoted by c , called the *exhaust velocity*. In this example, the mass flow rate and exhaust velocity are assumed to be constant.

DEFINITION 2.2 The velocity at which the exhaust stream exits the rocket is called the **exhaust velocity**. This velocity is measured relative to the rocket engine.

How can we express the thrust in terms of the mass flow rate and exhaust velocity? Let's start by applying Newton's Second Law to the exhaust of the rocket. The only force acting on the exhaust is the thrust, which comes from the rocket pushing on the exhaust. We will also assume that the thrust force on the exhaust points in the positive x -direction, so we will only need to consider the change in linear momentum along the x -axis. We can write Newton's Second Law as:

$$F = \frac{dp}{dt} \quad (1)$$

where F is the thrust and p is the linear momentum of the exhaust in the x -direction.

Let's consider a chunk of exhaust as it is accelerated out of the rocket engine. The momentum of the chunk changes as it is accelerated by the thrust force. We'll assume that the exhaust has zero momentum before it is accelerated, which is basically saying that the initial velocity of the exhaust is zero. We can write this as $p_i = 0$, where p_i is the initial momentum. Somehow, the rocket engine accelerates the exhaust to a speed c . The final momentum of the exhaust, p_f , is the product of the mass of the exhaust chunk, Δm , and the exhaust velocity, c :

$$p_f = \Delta m c \quad (2)$$

Therefore, the change in linear momentum of a chunk of exhaust is:

$$\begin{aligned}\Delta p &= p_f - p_0 \\ &= \Delta m c\end{aligned}\quad (3)$$

By Newton's Second Law, the change in exhaust momentum per unit time is equal to the thrust force acting on the exhaust. We can write the change in momentum of the exhaust per unit time as:

$$\begin{aligned}\frac{\Delta p}{\Delta t} &= \frac{\Delta m}{\Delta t} c \\ &= \dot{m} c\end{aligned}\quad (4)$$

where Δt is the time period over which the thrust force acts on the exhaust. This is equal to the time it takes the exhaust to pass through the rocket engine. We can recognize that the mass of the exhaust chunk divided by the time it takes for the exhaust to pass through the rocket is equal to the mass flow rate, \dot{m} .

Now we can compute the thrust force, which, assuming there are no other forces acting on the exhaust, is equal to $\Delta p / \Delta t$:

$$F = \dot{m} c \quad (5)$$

The thrust of a rocket that continuously expels exhaust is given by the product of the mass flow rate and the exhaust velocity. Remember that the thrust force that propels the exhaust out of the rocket engine is equal to the force that accelerates the rocket vehicle. Let's do an example!

Example 1:

We can apply equation 5 to our previous example about tennis balls. If we want to produce a thrust of 100 N, how many tennis balls does the astronaut need to serve per second? The mass of a tennis ball, m_b , is 59 g. Assume that the exhaust velocity, c , of the tennis balls is 235 km/hr (which is equivalent to 65 m/s).

Let's begin by calculating the required mass flow rate of tennis balls using equation 5:

$$\dot{m} = \frac{F}{c} = \frac{100 \text{ N}}{65 \text{ m/s}} = 1.54 \text{ kg/s} \quad (1.1)$$

Now let's calculate the serve rate, \dot{n} , which is the number of tennis balls served per second. The serve rate is equal to the mass flow rate divided by the mass of each tennis ball:

$$\dot{n} = \frac{\dot{m}}{m_b} = \frac{1.54 \text{ kg/s}}{0.057 \text{ kg}} = 27 \text{ tennis balls/second} \quad (1.2)$$

The astronaut needs to serve 27 million tennis balls per second to produce a thrust of 100 N, which is equivalent to the weight of 5 pounds. I think we're going to need a better way to produce thrust!

Okay, okay, I know the tennis ball example is getting a little old. How much thrust do REAL rockets produce? Well, it depends on the type of rocket and the mission goal. Chemical rockets typically produce $10^2 - 10^7$ Newtons of thrust. The first stage of a launch vehicle, which uses chemical rockets, typically produces at least one million Newtons of thrust force. The chemical rocket that was used to lift the Lunar Module off the surface of the Moon produced 16,000 N of thrust. Electrically-powered rockets, on the other hand, produce less than 1 N of thrust. We will investigate why chemical and electrical rockets have such different performance in later lectures.

Example 2:

Let's estimate the thrust of the Space Shuttle Main Engine (SSME). The mass flow rate, \dot{m} , is 470 kg/s and the exhaust velocity, c , is 4440 m/s.

$$F = \dot{m}c = (470 \text{ kg/s})(4440 \text{ m/s}) = 2.01 \times 10^6 \text{ N} \quad (2.1)$$

The SSME produces approximately two million Newtons of thrust, which is equivalent to the weight of a blue whale!



Figure 3: One Space Shuttle Main Engine produces a thrust force equivalent to the weight of a blue whale! Blue whales can be up to 100 ft in length.

SPECIFIC IMPULSE

Now that we have an expression for rocket thrust, we can define some other performance parameters for rockets. One of the most important performance metrics for rockets is the *specific impulse*, which is considered the "fuel efficiency" of a rocket.

DEFINITION 2.3 The **specific impulse** is defined as the total impulse delivered by the rocket engine divided by the total weight of propellant expended. The units for specific impulse, denoted by the symbol I_{sp} , are seconds.

Let's derive an expression for the specific impulse. The total impulse, I_T , is the force delivered by the rocket integrated over time:

$$I_T = \int_0^{t_b} F(t) dt \quad (6)$$

where t_b is the total time during which the rocket fires, which is also called the "burn time." Note that the thrust force, F , can be a function of time.

To compute the specific impulse, we need to calculate the total weight of expended propellant, W_T . This can be found by integrating the mass flow rate of the exhaust over the burn duration:

$$W_T = g \int_0^{t_b} \dot{m}(t) dt \quad (7)$$

where g is the acceleration due to gravity on Earth's surface. Note that the mass flow rate can be a function of time.

Now we can write an expression for the specific impulse:

$$I_{sp} = \frac{I_T}{W_T} = \frac{\int_0^{t_b} F(t) dt}{g \int_0^{t_b} \dot{m}(t) dt} \quad (8)$$

We can simplify this expression by assuming that the thrust and mass flow rate are constant during the rocket burn:

$$I_{sp} = \frac{F}{\dot{m}g} \quad (9)$$

Given our expression for rocket thrust, see equation 5, we can express the specific impulse as:

$$I_{sp} = \frac{c}{g} \quad (10)$$

Later in this lecture, we'll learn that higher specific impulse results in higher fuel efficiency. To achieve a high specific impulse, we need to design a rocket with a high exhaust velocity.



Figure 4: Fuel efficiency is measured in seconds?? It sounds strange, but yes, fuel efficiency for rockets is measured in seconds. You can think of the I_{sp} as the number of seconds that 1 kilogram of propellant can be used to produce 9.8 Newtons of thrust. The higher the I_{sp} , the longer the rocket can operate per kilogram of propellant.

DERIVATION OF THE IDEAL ROCKET EQUATION

In this section, we will derive a fundamental equation of rocket propulsion, called *The Rocket Equation*. This equation will allow us to calculate how much propellant we need to accelerate a vehicle by a specified change in velocity, Δv . It will also help us understand the relationship between fuel efficiency and specific impulse.

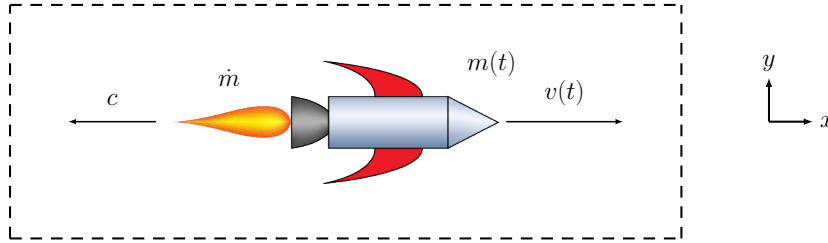


Figure 5: A rocket-propelled vehicle that continuously expels exhaust. The dashed box defines the system, which contains the vehicle and all of the exhaust that has been expelled. You can imagine that the dashed box is a flexible container that expands so it always contains the vehicle and all of the exhaust.

Let's consider the total momentum of the vehicle and the exhaust. We can think of the vehicle and the exhaust as a system, as shown in Figure 6. We will assume that no external forces, such as gravity, are acting on the system. This allows us to assume that the total momentum of the system is constant, by Newton's Second Law. In other words, the rate of change of the total momentum of the system is zero:

$$\left. \frac{dp}{dt} \right|_{total} = \left. \frac{dp}{dt} \right|_{vehicle} + \left. \frac{dp}{dt} \right|_{exhaust} = 0 \quad (11)$$

The momentum of the vehicle is mv , where the mass and velocity of the vehicle are functions of time. We can calculate the time rate of change of the vehicle momentum by differentiating mv with respect to time:

$$\left. \frac{dp}{dt} \right|_{vehicle} = \frac{d}{dt}(mv) = \frac{dm}{dt}v + m \frac{dv}{dt} \quad (12)$$

The rate of change of the mass of the vehicle, dm/dt , is related to the mass flow rate of the exhaust, \dot{m} by the following expression:

$$\frac{dm}{dt} = -\dot{m} \quad (13)$$

The vehicle loses mass at the same rate that the exhaust plume gains mass. The time rate of change of the momentum of the vehicle is:

$$\left. \frac{dp}{dt} \right|_{vehicle} = m \frac{dv}{dt} - \dot{m}v \quad (14)$$

Now let's consider the time rate of change of momentum of the exhaust plume. The total momentum of the exhaust, p_e , is:

$$p_e = m_e v_e \quad (15)$$

where m_e is the mass of the exhaust and v_e is the velocity of the exhaust relative to a stationary observer. To obtain the time rate of change of momentum of the exhaust, let's differentiate equation 15 with respect to time:

$$\left. \frac{dp}{dt} \right|_{\text{exhaust}} = \frac{dm_e}{dt} v_e + m_e \frac{dv_e}{dt} \quad (16)$$

The rate of change of the mass of the exhaust plume is \dot{m} . Also, the velocity of the exhaust, v_e , is constant for a particular section of the exhaust plume. Once the exhaust leaves the vehicle, it is not acted upon by any forces, so its speed remains constant. Therefore we can conclude that $dv_e/dt = 0$. The time rate of change of the momentum of the exhaust is:

$$\left. \frac{dp}{dt} \right|_{\text{exhaust}} = \dot{m} v_e \quad (17)$$

What is v_e ? It's the velocity of the exhaust relative to a stationary observer, which depends on the velocity of the rocket!

Consider a stationary observer watching the vehicle fly through space at a velocity v in the positive x-direction. The observer also sees the exhaust leaving the vehicle. The exhaust is expelled in a direction opposite of the vehicle's motion at a velocity, c . Remember that c is the exhaust velocity, which is measured *relative to the vehicle*. If the vehicle were stationary, then the net velocity of the exhaust would be $-c$, since we assume that the exhaust travels in the negative x-direction. But we know that the vehicle is not stationary, it's moving in the positive x-direction at a velocity, v . To a stationary observer, the net velocity of the exhaust leaving the rocket at a particular moment in time is $v - c$. So we can write equation 17 as

$$\left. \frac{dp}{dt} \right|_{\text{exhaust}} = \dot{m}(v - c) \quad (18)$$

Now we can combine equation 14 and equation 18 to obtain the time rate of change of the total momentum of the system:

$$\begin{aligned} \left. \frac{dp}{dt} \right|_{\text{total}} &= m \frac{dv}{dt} - \dot{m}v + \dot{m}(v - c) \\ &= m \frac{dv}{dt} - \dot{m}c \end{aligned} \quad (19)$$

By Newton's Second Law, the time rate of change of the momentum of the system is zero:

$$0 = m \frac{dv}{dt} - \dot{m}c \quad (20)$$

Using equation 13, we can write equation 20 as:

$$m \frac{dv}{dt} = - \frac{dm}{dt}c \quad (21)$$

We can cancel out the dt 's to obtain a first order, linear differential equation:

$$m dv = -dm c \quad (22)$$

To solve this equation, we need to rearrange it so the m variables are on one side of the equation and the v variables are on the other.

$$\frac{dm}{m} = - \frac{dv}{c} \quad (23)$$

Now we can integrate both sides. On the lefthand side we will integrate from the initial mass of the vehicle, m_0 , to the final mass of the vehicle, m_f . On the righthand side we will integrate from the initial velocity of the vehicle, v_0 , to the final velocity of the vehicle, v_f .

$$\begin{aligned} \int_{m_0}^{m_f} \frac{dm}{m} &= - \int_{v_0}^{v_f} \frac{dv}{c} \\ \ln(m) \Big|_{m_0}^{m_f} &= - \frac{1}{c} v \Big|_{v_0}^{v_f} \\ \ln\left(\frac{m_f}{m_0}\right) &= - \frac{v_f - v_0}{c} \end{aligned} \quad (24)$$

Let's call the change in velocity of the vehicle, $v_f - v_0$, *delta-v*.

DEFINITION 2.4 **Delta-v** is the change in velocity of a rocket-propelled vehicle. It is typically denoted by the symbol Δv and has units of m/s.

We can rewrite equation 24 using $\Delta v = v_f - v_0$. We will also assume that the final mass of the rocket is equal to the initial mass minus the mass of propellant expended, m_p .

$$\ln\left(\frac{m_0 - m_p}{m_0}\right) = - \frac{\Delta v}{c} \quad (25)$$

Differential equation? Don't worry, you're not expected to know what differential equations are or how to solve them! Just follow along with the derivation and trust your calculus instincts :) The end result will be what's most important for the course.

Finally, let's solve for m_p/m_0 to obtain the *Ideal Rocket Equation*:

$$\frac{m_p}{m_0} = 1 - \exp(-\Delta v/c) \quad (26)$$

Equation 26 is called the Ideal Rocket Equation because it can only be applied to idealized scenarios where there are no external forces acting on the rocket-exhaust system. Note that launching from the surface of the Earth involves external forces (gravity!) so the Ideal Rocket Equation can't be used to accurately estimate the propellant needed to get to space! We will learn more about this next lecture.

The Ideal Rocket Equation allows us to calculate the mass of propellant required to achieve a desired change in vehicle velocity, Δv , given a constant exhaust velocity, c , and an initial vehicle mass, m_0 . Notice that the larger the initial vehicle mass, the more propellant mass is needed to achieve some Δv . This makes sense, because it takes more energy to move a heavier object. Also the larger the required Δv , the more propellant mass is needed. This also makes sense because more energy is needed to create a larger increase in the kinetic energy of the vehicle.

We can also notice that the propellant mass is inversely related to the exhaust velocity. This means that a higher exhaust velocity results in a lower required propellant mass. This relationship is less intuitive, but we can understand it by considering a unit of exhaust of fixed mass. The faster the unit of exhaust leaves the vehicle, the more momentum it imparts to the vehicle. This means more thrust and thus more acceleration. Therefore, the faster the propellant is expelled from the vehicle, the more quickly the vehicle accelerates, so less propellant mass is needed to achieve a given Δv .

We can also write the Ideal Rocket Equation in terms of the specific impulse:

$$\frac{m_p}{m_0} = 1 - \exp\left(-\frac{\Delta v}{g I_{sp}}\right) \quad (27)$$

Now we can more clearly see the relationship between I_{sp} and fuel efficiency. The higher the I_{sp} , the less propellant mass needed to achieve a given Δv .