INTRODUCTION TO ROCKET PROPULSION

L3: Gravity and Orbits

What are orbits and how do we describe them mathematically?
In this lecture, we will learn about orbits! We'll start by reviewing how to represent gravitational forces using vector notation. Next, we'll consider circular orbits, which are the simplest type of orbit. Many satellites are placed in circular orbits around the Earth. Next, we'll investigate the motion of the planets, which can be described by Kepler's Laws. The planets travel around the Sun in ellipses, so we'll learn how to mathematically describe elliptical orbits.

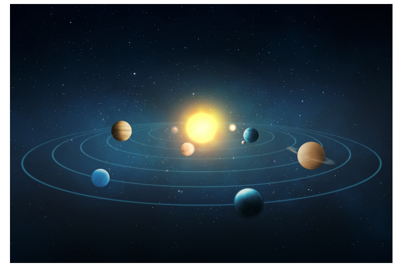

Figure 1: Artist's rendering of the Solar System planets orbiting the Sun.

LEARNING GOALS:

1. Calculate the gravitational force that arises between two masses using vector notation.
2. Draw a diagram of a satellite in a circular orbit, including the radius and velocity vectors of the satellite.
3. Compute the velocity of a circular orbit of a specified radius. Use the velocity to compute the orbital period.
4. Use Kepler's Laws to describe the characteristics of elliptical orbits.
5. Write the equation for the specific energy of an orbit.

GRAVITY

Gravity is the most apparent force in nature because we can clearly observe its effects. When we accidentally drop something, like an apple, for example, we know it's gravity that makes the object fall to the ground. When we ride in a car down a big hill, we know it's gravity that makes the car go faster and faster. Based on what we observe in our everyday lives, we might conclude that gravity makes things fall down and keeps us tethered to Earth's surface. This is true, but *gravity* is more than that!

DEFINITION 3.1 **Gravity** is the mutual interaction that arises between bodies that have mass. It is an attractive force that pulls massive objects towards one another.

Weight

Let's start by considering the gravity that we know best, the force that gives objects their weight. For example, the weight, W, of an object of mass m is given by the following expression:

$$W = mg \tag{1}$$

where g is the acceleration due to gravity on Earth's surface. The value of g depends on what altitude, the height above sea-level, you're located at. At sea-level, $g = 9.81\,\mathrm{m/s^2}$.

The force of gravity keeps objects firmly on the ground. For an object to leave Earth's surface, even if only for a brief moment, we need to exert a force that is larger than the weight of the object. For example, most of us have probably thrown a ball into the sky. We were able to do this because we could exert an upward force on the ball larger than its weight. Ultimately the ball fell back to the ground, but for a brief moment it flew up above Earth's surface. Likewise, rockets on the launch pad need to produce a thrust force that exceeds the total weight of the vehicle for the rocket to even lift off the ground. To reach orbit, rockets must continuously apply a thrust force greater than the weight of the vehicle as the vehicle ascends through Earth's atmosphere.

General Definition of Gravity

What if we are not on Earth's surface? How do we describe gravity then? Let's consider two masses, m_1 and m_2, in the xy-plane, shown in Figure 2. The vector $\vec{r}_1$ describes the location of the first mass, relative to the origin of the coordinate system. Likewise, the vector $\vec{r}_2$ describes the location of the second mass, relative to the origin. Finally, the vector $\vec{r}_{12}$ describes the location of the second mass relative to the first mass. Note that there are two ways to get from the origin to the second mass. One can follow along $\vec{r}_2$ directly, or one can follow $\vec{r}_1$ and then follow $\vec{r}_{12}$. Therefore we can say that $\vec{r}_1 + \vec{r}_{12} = \vec{r}_2$.

We know that an attractive force will arise between these two masses, shown in Figure 3. The first mass will exert a force, $\vec{F}_{12}$, on the second mass, pulling the second mass towards itself. The second mass will exert an equal and opposite force, $\vec{F}_{21}$ on the first mass. By Newton's Third Law, we know that $\vec{F}_{12} = -\vec{F}_{21}$.

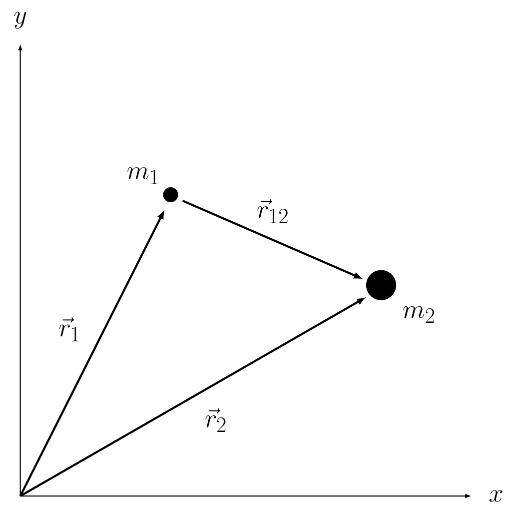

Figure 2: Locations of two masses in the xy-plane.

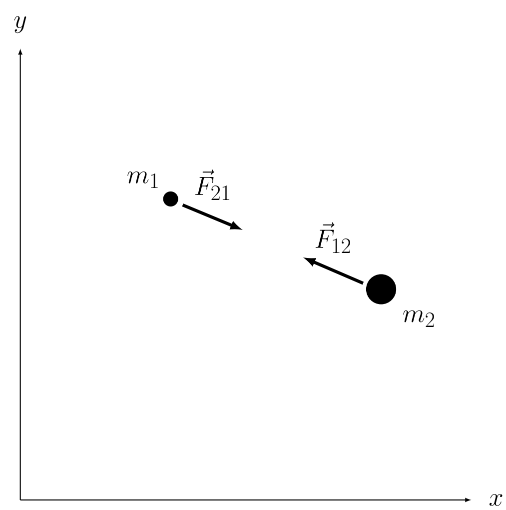

Figure 3: Attractive forces due to gravity between two masses in space. $\vec{F}_{12}$ has the same magnitude but points in the opposite direction of $\vec{F}_{21}$.

We can express the force due to gravity on the second mass by the presence of the first mass using the following mathematical formula:

$$\vec{F}_{12} = -\frac{G\, m_1\, m_2}{(r_{12})^2}\ \hat{\mathbf{r}}_{\mathbf{12}} \tag{2}$$

where G is the Gravitational constant, which is equal to $6.67 \times 10^{-11}\ \mathrm{m}^3/\mathrm{kg}\cdot\mathrm{s}^2$. The distance between the two masses is given by r_{12}, which is the magnitude of the vector $\vec{r}_{12}$. The vector $\hat{\mathbf{r}}_{\mathbf{12}}$ is a *unit vector* that points from the first mass to the second mass, and is defined as:

$$\hat{\mathbf{r}}_{\mathbf{12}} = \frac{\vec{r}_{12}}{r_{12}} \tag{3}$$

Notice that the unit vector has a magnitude equal to 1. When we multiply something by a unit vector, we don't change the magnitude because anything multiplied by 1 is itself. Unit vectors are used because they allow us to specify a direction.

The magnitude of the gravitational force on the second mass due to the presence of the first mass is:

$$F_{12} = \frac{G\, m_1\, m_2}{(r_{12})^2} \tag{4}$$

Just knowing the magnitude isn't enough information to understand how the force will affect the motion of the second mass. We need to know what *direction* the force acts in! This is where the unit vector comes in. We know the force should point towards the first mass, because gravity is an attractive force. That's why we use the unit vector to specify the direction pointing from the first mass to the second mass. We also put a minus sign to indicate that the force is attractive, in that it points from the second mass to the first mass.

We can observe from equation 2 that the magnitude of the gravitational force between the two masses decreases as the distance between the two objects increases. The force also decreases as the masses of the objects decrease. As a rocket ascends through Earth's atmosphere, it gets farther and farther away from Earth's center. Therefore, we should expect the gravitational force pulling the rocket towards the center of the Earth to decrease with increasing altitude.

CIRCULAR ORBITS

Once a rocket escapes Earth's surface, it usually enters an *orbit* around the Earth.

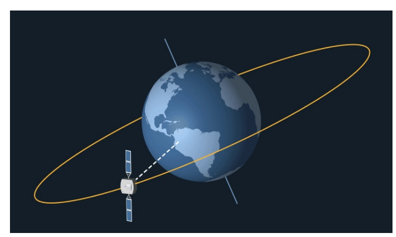

Figure 4: A satellite in a circular orbit around the Earth. It travels along this path again and again.

> DEFINITION 3.2 An **orbit** is a stable, closed trajectory around a massive body, such as the Earth.

The simplest type of orbit is a *circular orbit*. The path of a circular orbit is a circle, which is shown in Figure 5. The diagram shows a massive body, with mass M, at the center of the orbit. A much smaller body, with mass m, is shown following a circular path around the massive body. This diagram could be used to represent a satellite orbiting the Earth or perhaps the Earth orbiting the Sun.

The gravitational force acting on the smaller body is:

$$\vec{F}_G = -\frac{\mu m}{r^2}\, \hat{\boldsymbol{r}} \tag{5}$$

Where μ is the *gravitational parameter*, which is the product of the Gravitational constant and the mass of the larger body: $\mu = GM$. The gravitational parameter is used to make computations easier.

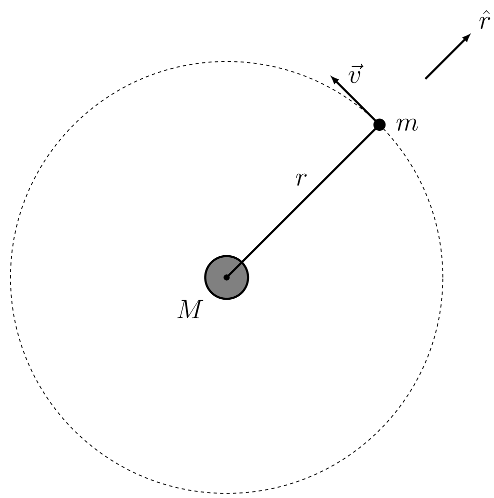

Figure 5: Diagram of a circular orbit. The path of the orbit is drawn with a dashed grey line.

The smaller body follows a circular path, which is a curved trajectory. Recall that the centripetal acceleration, $\vec{a}_c$, of a body traveling along a circular path is given by the following expression:

$$\vec{a}_c = -\frac{v^2}{r}\ \hat{\boldsymbol{r}} \tag{6}$$

where v is the magnitude of the velocity of the body, also called its speed, and r is the radius of the circular path. Note that the centripetal acceleration points towards the center of the circular path.

We can write Newton's Second Law for the smaller body:

$$\begin{aligned} \vec{F}_G &= m\vec{a}_c \\ -\frac{\mu m}{r^2}\ \hat{\boldsymbol{r}} &= -\frac{mv^2}{r}\ \hat{\boldsymbol{r}} \end{aligned} \tag{7}$$

Observe that the gravitational force is balanced by the centripetal acceleration. Since both vectors point along the radial direction, we can remove the unit vectors and solve equation 7 for the speed of the smaller body:

$$v = \sqrt{\frac{\mu}{r}} \tag{8}$$

The velocity of a body in a circular orbit is *constant in magnitude*. The direction of the velocity is always perpendicular to the radial vector that points from the central body to the smaller body.

The magnitude of the velocity only depends on the mass of the central body and the orbital radius. The velocity increases as the mass of the central body increases. The velocity also increases as the orbital radius decreases. Note that the velocity does not depend on the mass of the smaller body because the m's canceled out in equation 7.

A smaller body orbits a massive body along a circular path with a radius, r, if and only if:

1. The smaller body is at an orbital altitude such that the distance between it and the center of the massive body is r.

2. The magnitude of the velocity of the smaller body is exactly that prescribed by equation 8.

3. The direction of the smaller body's velocity is always perpendicular to the radial vector that points from the center of the massive body to the smaller body. In other words, the velocity direction is always tangential to the circular path.

If any of these items are not satisfied, the body will not travel in a circular orbit with a radius r. Instead, the body will probably travel in an elliptical orbit.

Common Satellite Orbits

There are two major categories of orbits around the Earth that most satellites are placed in:

1. **Low Earth Orbit (LEO)** - LEO orbits range from 100-1000 km in altitude. For example, the International Space Station (ISS) is 180 km above Earth's surface. Satellites in LEO orbits complete several revolutions around the Earth per day. This means that LEO satellites travel over many different regions of the Earth each day.

2. **Geostationary Orbit (GEO)** - GEO orbits form a narrow band around 35,786 km in altitude. GEO orbit is special because its period is exactly 1 day. This means that satellites stay above the same region of the Earth at all times.

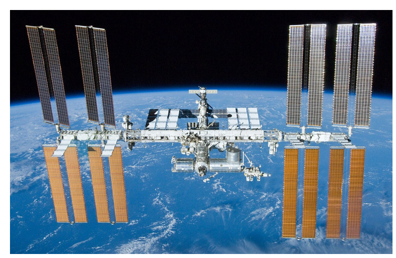

Figure 6: The International Space Station (ISS). Did you know that the ISS is as big as an American football field?

Example 1:

Let's do an example with circular orbits! The ISS follows a circular orbit around the Earth at an altitude of 180 km. What is the orbital velocity of the ISS? How many revolutions around the Earth does the ISS complete per day?

We can calculate the orbital velocity using equation 8. Note that the orbital radius, r, is equal to Earth's radius, R_E, plus the orbital altitude, h.

$$v = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{\mu}{R_E + h}} \tag{1.1}$$

Let's plug in some numbers. The gravitational parameter for Earth is $3.986 \times 10^{14}\ \mathrm{m^3/s^2}$ and the Earth's equatorial radius is 6378 km.

$$\begin{aligned} v &= \sqrt{\frac{3.986 \times 10^{14}\,\mathrm{m^3/s^2}}{6378\mathrm{km} + 180\mathrm{km}}} \\ &= 7.8\,\mathrm{km/s} \end{aligned} \tag{1.2}$$

Wow, that's fast!

To figure out how many revolutions the ISS completes per day, we need to calculate the time it takes to complete one revolution. We know the velocity of the ISS, so if we can get the distance traveled during one revolution, we can calculate the time for one revolution. The orbit is a circle of radius r, and the circumference of the orbit is $2\pi r$. Therefore, the distance traveled during one revolution is $2\pi r$. Now we can calculate the orbital period, $\mathbb{T}$:

$$\begin{aligned} \mathbb{T} &= \frac{2\pi r}{v} \\ &= \frac{2\pi(6378\mathrm{km} + 180\mathrm{km})}{7.67\,\mathrm{km/s}} \\ &= 5280\,\mathrm{s} \\ &= 1.46\,\mathrm{hours} \end{aligned} \tag{1.3}$$

We can divide the time in one day by the orbital period to get the number of revolutions per day. The result is 16.3 revolutions per day. This means that astronauts on board the ISS see approximately 16 sunrises and 16 sunsets per day!

MOTION OF SOLAR SYSTEM BODIES

Circular orbits are a good approximation for the orbits of most satellites and even many planets. In actuality, the planets orbit the Sun in slight ellipses, not circles. Comets, in particular, orbit the Sun in highly elliptical orbits. Elliptical orbits are also used to change the orbit of a spacecraft, which we will learn about next lecture. In this section, we'll learn about the motion of Solar System bodies, which can be succinctly described by Kepler's Laws.

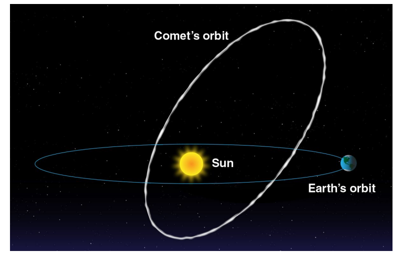

Figure 7: Comets travel on highly elliptical orbits that bring them very close to the Sun at one end and to the outer reaches of the Solar System on the other end.

Kepler's Laws, listed below, concisely describe the motion of Solar System planets around the Sun. These laws also apply to planets orbiting other stars.

Kepler's Laws

1. All planets move in elliptical orbits, with the Sun at one focus.
2. A line that connects a planet to the Sun sweeps out equal areas in equal times.
3. The square of the period of any planet's orbit is proportional to the cube of the semi-major axis of its orbit.

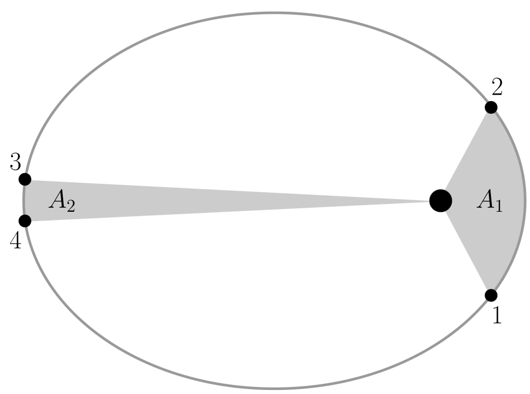

Figure 8: An illustration of Kepler's Second Law, which is best described using an animation - check out this **YouTube video**!

The first law states that all planets move in elliptical orbits around the Sun. We'll learn how to mathematically describe elliptical orbits later on in this lecture.

The second law states that a line connecting the Sun to a planet sweeps out equal areas in equal times, illustrated by Figure 8. The Sun is shown as a large black dot located at the rightmost foci of the ellipse. The planet is the small black dot that travels along the perimeter of the ellipse. Let's consider the area A_1 that is swept out as the planet moves from location 1 to location 2. We will assume that it takes Δt seconds for the planet to move from location 1 to 2. Figure 8 also shows another area, A_2, that is swept out by the planet as it travels from location 3 to location 4. If we assume that it takes Δt seconds for the planet to move from location 3 to 4, then we can conclude that $A_1 = A_2$ by Kepler's Second Law.

What does this tell us about the planet's velocity as it orbits the star? Kepler's Second Law is really a statement about the conservation of angular momentum, which tells us how the velocity of the planet changes along its orbit. Consider the planet's motion from location 1 to 2 in comparison to its motion from location 3 to 4. The planet travels a longer distance along its path as it sweeps out A_1 than that when it sweeps out A_2. Let's assume that the planet sweeps out A_1 and A_2 in the same amount of time, Δt. Velocity is distance divided by time, so as the planet sweeps out A_1, it travels a long distance over a period Δt. In contrast, as the planet sweeps out A_2, it travels a short distance over the same period of time Δt. Therefore the velocity of the planet as it sweeps out A_1 is much higher than that when it sweeps out A_2. *Kepler's Second Law tells us that the planet travels fastest when it is closest to the Sun and slowest when it is farthest from the Sun.*

Finally, Kepler's Third Law tells us about the relationship between the period of a planet's orbit to the size of the orbit. We will see how express this law mathematically during the lab!

ELLIPTICAL ORBIT PARAMETERS

In this section, we'll learn how to mathematically describe elliptical orbits. Figure 9 shows a planet of mass m traveling along an elliptical path around the Sun. Notice that the ellipse has two foci: one on the left, which is empty, and one on the right, which is the location of the Sun. The total width of the ellipse is $2a$, where a is the *semi-major axis*. The total height of the ellipse is $2b$, where b is the *semi-minor axis*. The *eccentricity*, e, describes how "ellipitical" an ellipse is:

$$e = \sqrt{1 - \frac{b^2}{a^2}} \tag{9}$$

The minimum possible value of eccentricity is zero. In this case, $a = b$, which means we have a circle. The maximum possible value of eccentricity is 1. In this case, $a \gg b$, which means we have an infinitely wide, infinitely short ellipse. Most of the Solar System planets follow elliptical orbits with $e < 0.1$, which means that their orbits are approximately circular.

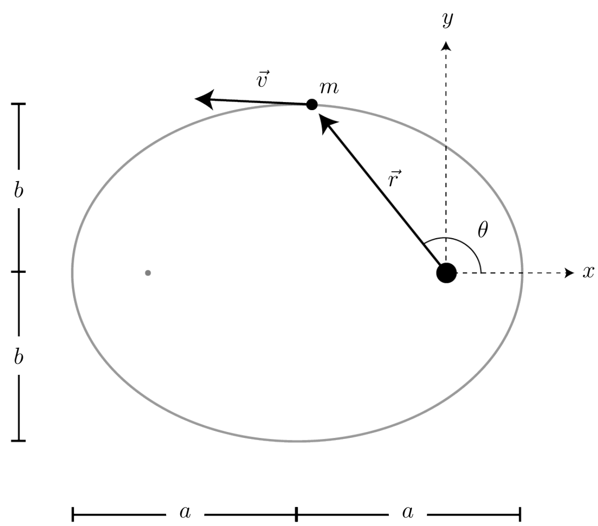

Figure 9: Diagram of an elliptical orbit.

Now let's consider the motion of the planet. Notice that the x-y coordinate system is centered at the location of the Sun. We can draw a position vector, $\vec{r}$, from the Sun to the planet, as shown in Figure 9. The angle between the positive x-axis and the planet's position vector is called the *argument of perigee* and is denoted by θ. Finally, we can draw the planet's velocity vector, $\vec{v}$, which is always tangent to the curve of the ellipse.

Notice that the position vector and velocity vector are not perpendicular to each other like they were for circular orbits. In fact, the relative angle between the position and velocity vectors changes as the planet orbits the Sun. There are only two places where the position vector and velocity vector are perpendicular: *perigee* and *apogee*.

DEFINITION 3.3 The position of the planet when it is closest to the Sun is called **perigee**. When the *argument of perigee* is zero, the planet is at *perigee*.

DEFINITION 3.4 The position of the planet when it is farthest from the Sun is called **apogee**. When the *argument of perigee* is 180°, the planet is at *apogee*.

Figures 10 and 11 show a planet at *perigee* and *apogee*, respectively. Notice from Figures 10 and 11 that $r_p + r_a = 2a$.

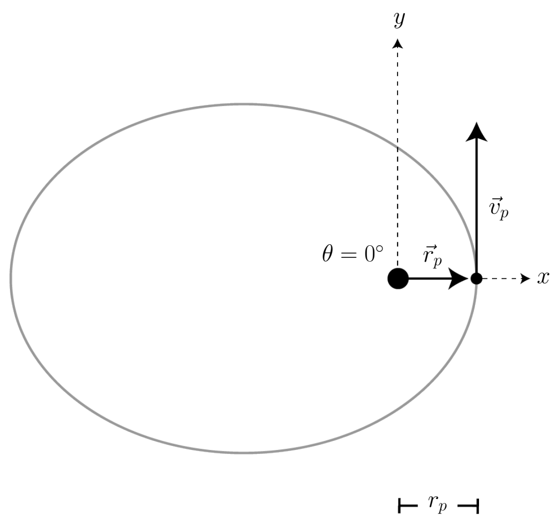

Figure 10: Perigee of an elliptical orbit. The distance between the planet and the Sun is r_p, which is called the *perigee radius*. The velocity is $\vec{v}_p$, which is called the *perigee velocity*

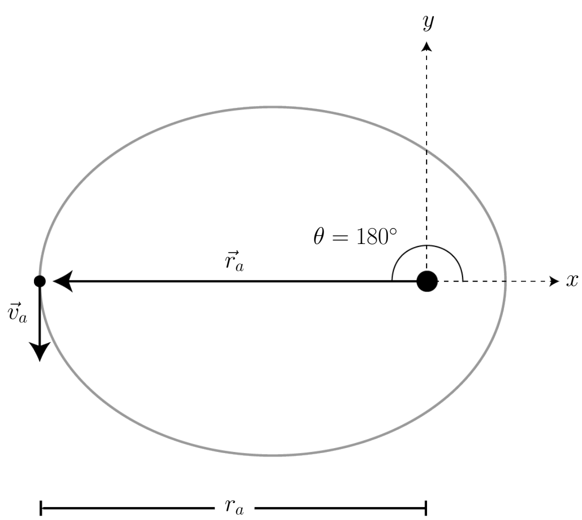

Figure 11: Apogee of an elliptical orbit. The distance between the planet and the Sun is r_a, which is called the *apogee radius*. The velocity is $\vec{v}_a$, which is called the *apogee velocity*.

ELLIPTICAL ORBIT DYNAMICS

Now that we have specified the parameters of elliptical orbits, we can consider the motion of the planets according to Kepler's Second and Third Laws. Recall that Kepler's Second Law is a statement about the conservation of angular momentum of the system. We can write the angular momentum, $\vec{L}$, of the planet as:

$$\vec{L} = m\,\vec{r} \times \vec{v} \tag{10}$$

When considering planetary orbits, we usually consider the *specific angular momentum*:

$$\vec{h} = \vec{r} \times \vec{v} \tag{11}$$

The specific angular momentum, h, is the angular momentum divided by the mass of the planet. If we assume that there are no external forces acting upon the Sun and planet, then the angular momentum must be constant over time:

$$\frac{d\vec{h}}{dt} = 0 \tag{12}$$

This means that both the magnitude and direction of the specific angular momentum are constant. The magnitude of the specific angular momentum is:

$$h = rv_{\perp} \tag{13}$$

where $v_{\perp}$ is the velocity component that is perpendicular to the radius vector at some point along the orbit. For elliptical orbits, the radius and velocity vectors are perpendicular to each other at two locations: perigee and apogee. Since the angular momentum is constant, we can write that:

$$h = r_p v_p = r_a v_a \tag{14}$$

where r_p is the perigee radius, v_p is the velocity at perigee, r_a is the apogee radius, and v_a is the velocity at apogee.

Kepler's Equation

By applying the conservation of angular momentum, Newton's Second Law, and geometrical properties of ellipses, we can derive an equation that describes the shape of an elliptical orbit. The derivation is lengthy and involves quite a bit of vector mathematics, so we won't go into the details here. The result is called *Kepler's Equation*, which gives the orbital radius, r, as a function of the argument of perigee, θ:

$$r = \frac{h^2/\mu}{1 + e\cos\theta} \tag{15}$$

Notice that h, e, and μ are constant for a given ellipse. This equation provides $r(\theta)$, which describes the shape on an ellipse in polar coordinates.

Energy of Orbits

The total energy, E, of a spacecraft orbiting a planet is given by the sum of its kinetic energy and gravitational potential energy:

$$\mathbb{E} = \frac{1}{2}mv^2 - \frac{\mu m}{r} \tag{16}$$

When considering orbits, we typically use the *specific energy*, $\mathcal{E}$, which is the total energy divided by the mass of the spacecraft:

$$\mathcal{E} = \frac{1}{2}v^2 - \frac{\mu}{r} \tag{17}$$

In the lab, we'll use many of the equations from this section to analyze the dynamics of elliptical orbits in greater detail. We'll derive some expressions that are critical for space missions, such as sending a spacecraft to Mars!