INTRODUCTION TO ROCKET PROPULSION

L4: Orbit Transfers and Solar System Exploration

How do we send a spacecraft from Earth to Mars?
In this lecture, we will learn how rockets are used to change the orbits of spacecraft. A popular technique is called a Hohmann transfer, which uses an elliptical transfer orbit that connects one circular orbit to another. We'll learn how to calculate the change in velocity, and thus propellant required, to complete a Hohmann transfer. We'll also calculate the velocity required to escape the gravitational field of a massive body, such as the Earth. Spacecraft orbiting the Earth need to escape Earth's gravitational influence before they can travel on to Mars and beyond! Finally, we'll learn about other types of orbit transfers on a conceptual level.

Figure 1: The Mars 2020 Rover, named Perseverance, is on it's way to Mars! It launched from Earth's surface on July 30, 2020 and will land on Mars' surface on February 18, 2021. Click **here** to track its orbital path on NASA JPL's webpage.

Learning Goals:

1. Describe how Hohmann transfers work. Draw a diagram of the initial orbit, transfer orbit, and final orbit.
2. Calculate the change in velocity and propellant required to complete a Hohmann transfer.
3. Calculate the change in velocity required to escape the gravitational influence of a massive body.
4. Conceptually describe other types of orbit transfers.

ORBIT TRANSFERS

Hohmann Transfers

Let's consider the situation in which a spacecraft is in a circular orbit around Earth with a radius r_1. The spacecraft needs to get to a higher altitude circular orbit of radius r_2. How do we move the spacecraft from one orbit to the other? One of the most common ways to change circular orbits is with a *Hohmann transfer*. The path of a Hohmann transfer orbit is illustrated by the black curve in Figure 2.

Notice how the black curve is half of an ellipse that connects the two circular orbits. The perigee of the ellipse has a radius r_1, which is the radius of the initial circular orbit. The apogee of the ellipse has a radius r_2, which is the radius of the final circular orbit. To get onto the transfer orbit, the spacecraft must fire its rockets and accelerate by a change in velocity Δv_1. To get off the transfer orbit and enter the final circular orbit, the spacecraft must fire its rockets again and accelerate by a change in velocity Δv_2.

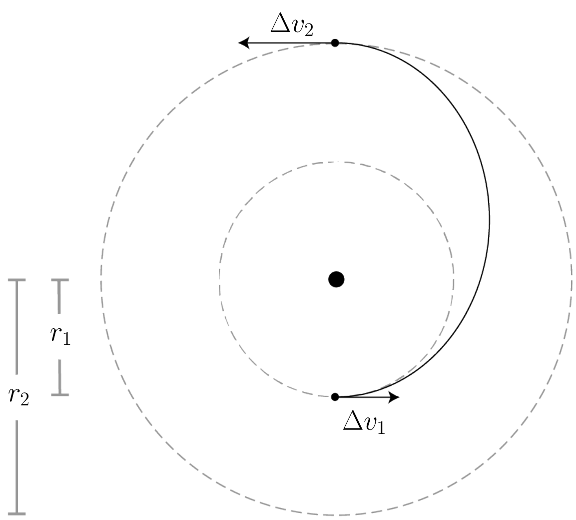

Figure 2: Hohmann transfer diagram. A spacecraft is initially in a circular orbit of radius r_1 around a massive body, denoted by the large black dot. The spacecraft uses a Hohmann transfer to reach a larger circular orbit around the massive body, which has a radius r_2.

The time it takes to complete a velocity change needs to be very short compared to the period of the orbit. A quick velocity change is called an *impulse maneuver*. Impulsive maneuvers allow us to assume that the spacecraft changes from one distinct orbit to another instantaneously. For example, in one moment the spacecraft is on its circular orbit of radius r_1. A moment later, the spacecraft is on its elliptical transfer orbit.

> DEFINITION 4.1 An **impulsive maneuver** consists of a short-duration firing of a spacecraft's rocket propulsion system. Typically an impulsive maneuver delivers high thrust over a short period of time, perhaps only a few minutes, which results in a large change in velocity.

To simplify the analysis of a Hohmann transfer, we need to assume that the spacecraft doesn't have time to move very far while the rockets are firing. During an impulsive maneuver, we can say that the position of the spacecraft is approximately constant and only the velocity changes.

When a spacecraft is located at some radius from Earth's center, it must travel at a particular velocity to be in circular orbit. This circular orbit velocity, v_{circ}, is given by the following equation:

$$v_{circ} = \sqrt{\frac{\mu}{r}} \tag{1}$$

Before the first impulsive maneuver in our Hohmann transfer example, the spacecraft is traveling at the following velocity, v_1:

$$v_{circ,1} = \sqrt{\frac{\mu}{r_1}} \tag{2}$$

After the impulsive maneuver, the spacecraft has a new velocity:

$$v_1 = v_{circ,1} + \Delta v_1 \tag{3}$$

We assume that the spacecraft is in the same position it was in before the rockets were fired, so that means it's still located at a distance r_1 from Earth's center. Now, the spacecraft is no longer be in a circular orbit because its position and velocity no longer satisfy equation 1. What type of orbit is the spacecraft in now? Well, assuming Δv_1 isn't too large, it will be in an ellptical orbit with a perigee radius $r_p = r_1$.

What is the apogee radius of this elliptical transfer orbit? Let's solve for r_a in terms of v_1. We can approach this problem by thinking about the energy of the spacecraft. Remember from the last lab that the specific energy, $\mathcal{E}$, of a spacecraft on an elliptical or circular orbit is given by the following expression:

$$\mathcal{E} = \frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{\mu}{2a} \tag{4}$$

where a is the semi-major axis of the ellipse, or radius of the circle. We can solve for a in terms of r and v:

$$a = \frac{1}{\frac{2}{r} - \frac{v^2}{\mu}} \tag{5}$$

Right after the first impulsive burn, $r = r_1$ and $v = v_1$. Therefore the semi-major axis of the transfer ellipse is:

$$a = \frac{1}{\frac{2}{r_1} - \frac{{v_1}^2}{\mu}} \tag{6}$$

We use the perigee radius and the semi-major axis to find the apogee radius:

$$\begin{aligned} r_a &= 2a - r_p \\ &= \frac{2}{\frac{2}{r_1} - \frac{{v_1}^2}{\mu}} - r_1 \end{aligned} \tag{7}$$

For a given initial radius, r_1, and a post-impulsive-maneuver velocity, v_1, we can determine the apogee radius, r_a of the spacecraft's elliptical transfer orbit.

For a Hohmann transfer, we know that the apogee radius of the transfer orbit should be the radius of the final circular orbit, r_2. What we don't know is how big Δv_1 needs to be to make the elliptical transfer orbit just the right size. We can solve for v_1 given that $r_p = r_2$ in equation 7:

$$v_1 = \sqrt{\frac{\mu}{r_1}\left(\frac{2\,r_2}{r_1 + r_2}\right)} \tag{8}$$

Finally we can solve for Δv_1:

$$\begin{aligned} \Delta v_1 &= v_1 - v_{circ,1} \\ &= \sqrt{\frac{\mu}{r_1}\left(\frac{2\,r_2}{r_1 + r_2}\right)} - \sqrt{\frac{\mu}{r_1}} \end{aligned} \tag{9}$$

Given an initial circular orbit radius and final circular orbit radius, we can use equation 9 to solve for the change in velocity required to enter the elliptical transfer orbit.

How do we get out of the transfer orbit? Without a second impulsive maneuver, the spacecraft will travel in an ellipse around the Earth over and over again. The spacecraft needs more kinetic energy to travel in a circle with radius r_2. Let's compare the the velocity of the final circular orbit with the velocity of the spacecraft at the apogee of its elliptical transfer orbit. The orbital velocity of the final circular orbit is:

Figure 3: Wait, there's more? All these equations are making me hungry...

$$v_{circ,2} = \sqrt{\frac{\mu}{r_2}} \tag{10}$$

The apogee velocity of the elliptical transfer orbit, v_2 is:

$$v_2 = \sqrt{\frac{\mu}{r_2}\left(\frac{2\,r_1}{r_1 + r_2}\right)} \tag{11}$$

Note that you can solve for this velocity using equation 6 and the relationship $r_a = 2a - r_p$.

We can calculate the change in velocity required for the second impulsive burn by subtracting the apogee velocity from the final circular velocity:

$$\begin{aligned} \Delta v_2 &= v_2 - v_{circ,2} \\ &= \sqrt{\frac{\mu}{r_2}} - \sqrt{\frac{\mu}{r_2}\left(\frac{2\,r_1}{r_1 + r_2}\right)} \end{aligned} \tag{12}$$

Now we have Δv_1 and Δv_2 in terms of the initial circular orbit radius, r_1, and the final circular orbit radius, r_2. The total change in velocity required to complete a Hohmann transfer is:

$$\begin{aligned} \Delta v &= \Delta v_1 + \Delta v_2 \\ &= \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\left(\frac{2\,r_2}{r_1+r_2}\right)}-1\right) + \sqrt{\frac{\mu}{r_2}}\left(1-\sqrt{\left(\frac{2\,r_1}{r_1+r_2}\right)}\right) \end{aligned} \tag{13}$$

Example 1:

Let's do an example! A satellite is dropped off in Low-Earth Orbit (LEO) by a launch vehicle. The initial altitude of the satellite is 180 km. Assume that it is in a circular orbit. The satellite needs to get to Geostationary Orbit (GEO), which has an altitude of 35,786 km. Calculate the total "delta-v" required for the satellite to change orbits using a Hohmann transfer.

We can start by calculating r_1 and r_2:

$$\begin{aligned} r_1 &= 6371\,\text{km} + 180\,\text{km} \\ &= 6551\,\text{km} \end{aligned} \tag{1.1}$$

$$\begin{aligned} r_2 &= 6371\,\text{km} + 35,786\,\text{km} \\ &= 42,157\,\text{km} \end{aligned} \tag{1.2}$$

We can calculate Δv_1 using equation 9. Let's start by computing the velocity of the inital circular orbit using equation 2. Remember that the gravitational parameter for the Earth is $\mu = 398,600\,\text{km}^3/\text{s}^2$.

$$\begin{aligned} v_{circ,1} &= \sqrt{\frac{\mu}{r_1}} \\ &= \sqrt{\frac{398,600\,\frac{\text{km}^3}{\text{s}^2}}{6551\,\text{km}}} \\ &= 7,800\,\text{m/s} \end{aligned} \tag{1.3}$$

Next, we can compute the perigee velocity of the elliptical transfer orbit using equation 8.

$$\begin{aligned} v_1 &= \sqrt{\frac{\mu}{r_1}\left(\frac{2\,r_2}{r_1 + r_2}\right)} \\ &= \sqrt{\frac{2(398,600\,\frac{\text{km}^3}{\text{s}^2})(42,157\,\text{km})}{(6551\,\text{km})(6551\,\text{km} + 42,157\,\text{km})}} \\ &= 10,263\,\text{m/s} \end{aligned} \tag{1.4}$$

The change in velocity required for the first impulsive burn is:

$$\begin{aligned} \Delta v_1 &= v_1 - v_{circ,1} \\ &= 10,263\,\text{m/s} - 7,800\,\text{m/s} \\ &= 2,463\,\text{m/s} \end{aligned} \tag{1.5}$$

Now let's calculate the change in velocity for the second impulsive burn. We can start by calculating the apogee velocity of the elliptical transfer orbit:

$$\begin{aligned} v_2 &= \sqrt{\frac{\mu}{r_2}\left(\frac{2\,r_1}{r_1 + r_2}\right)} \\ &= \sqrt{\frac{2(398,600\,\frac{\text{km}^3}{\text{s}^2})(6551\,\text{km})}{(42,157\,\text{km})(6551\,\text{km} + 42,157\,\text{km})}} \\ &= 1,595\,\text{m/s} \end{aligned} \tag{1.6}$$

The velocity for the final circular orbit is:

$$\begin{aligned} v_{circ,2} &= \sqrt{\frac{\mu}{r_2}} \\ &= \sqrt{\frac{398,600\,\text{km}^3/\text{s}^2}{42,157\,\text{km}}} \\ &= 3,075\,\text{m/s} \end{aligned} \tag{1.7}$$

The change in velocity for the second impulsive burn is:

$$\begin{aligned} \Delta v_2 &= v_{circ,2} - v_2 \\ &= 3,075\,\mathrm{m/s} - 1,595\,\mathrm{m/s} \\ &= 1,480\,\mathrm{m/s} \end{aligned} \tag{1.8}$$

Finally, the total required change in velocity of the satellite is:

$$\begin{aligned} \Delta v &= \Delta v_1 + \Delta v_2 \\ &= 2,463\,\mathrm{m/s} + 1,480\,\mathrm{m/s} \\ &= 3,943\,\mathrm{m/s} \end{aligned} \tag{1.9}$$

That's a sizeable change in velocity! It's nearly 50% of the change in velocity required to launch from the surface of the Earth to LEO, which we will investigate next lecture.

Example 2:

Let's calculate the propellant fraction required to complete the Hohmann transfer from example 1. Assume that the spacecraft has rocketsw with a specific impulse of 250 s.

Let's use the Ideal Rocket Equation to solve for the propellant mass fraction:

$$\begin{aligned} \frac{m_p}{m_0} &= 1 - \exp\left(-\frac{\Delta v}{g\, I_{sp}}\right) \\ &= 1 - \exp\left(-\frac{3,943\,\mathrm{m/s}}{(9.81\,\mathrm{m/s^2})(250\,\mathrm{s})}\right) \\ &= 0.80 \end{aligned} \tag{2.1}$$

The propellant fraction required to transfer from LEO to GEO is 80%. That's a lot of propellant!

Free Return Trajectories

Hohmann transfers can be used to raise and lower the orbits of satellites orbiting the Earth. They can also be used to travel from Earth's circular orbit around the Sun to Mars' circular orbit around the Sun. Can a Hohmann transfer be used to get to the Moon? Imagine that a spaceship initially orbits the Earth in a circle with radius r_1. We know that the Moon orbits the Earth in a circle as well, so why not set r_2 to the distance between the Earth and Moon? This is possible, but it requires a huge change in velocity. Is there a better way?

So far we have only considered what is called the *"two-body problem."* We have been looking at the orbits of a tiny spacecraft around a massive central body, such as the Earth. But what about the Moon? The Moon is massive enough to exert its own gravitational forces on a spacecraft that gets close enough. When spacecraft closely orbit the Earth, we can ignore the effect of the Moon. But when we are trying to go to the Moon itself, we need to consider the effect of its gravity too. This results in what is called the *"three-body problem."* The three bodies are the spacecraft, the Earth, and the Moon.

The three-body problem is significantly harder to solve. In fact, the trajectory of a spacecraft in the presence of two massive bodies can't be solved for analytically! That means we can't use pen and paper to write down some nice and neat equations like we did for the two-body problem. To solve the three-body problem, we need numerical methods and computer programs. Luckily, lots of people have been studying the three-body problem for a long time, so we can just take a look at the results of their hard work.

With the two-body problem, the possible closed orbits a spacecraft can have are either circular or elliptical. With the three-body problem, there are even more possible closed orbits and some have surpising shapes! This is because the spacecraft can orbit the Earth, the Moon, or both. One particular orbit is called the *"free return trajectory"*, which makes a "figure 8" shape between the Earth and Moon.

Closed orbits? If a spacecraft has enough energy to escape the gravity of the central body, its orbital path can be parabolic or hyperbolic. These trajectories are called open orbits.

Figure 4 shows the lunar free-return trajectory the Apollo astronauts took to the Moon. The spacecraft starts in the circular "parking orbit" around the Earth. Then, the rockets are fired to complete the "translunar injection" burn, which is an impulsive maneuver. The spacecraft coasts along the "figure 8" shape until it reaches the Moon. When the spacecraft arrives at the Moon, the rockets are burned again to slow the spacecraft down and enter a closed orbit around the Moon, which is not shown. When its time to go back to Earth, the rockets are fired to accelerate the spacecraft back into the free-return trajectory. The spacecraft coasts along the "figure 8" shape until it reaches the Earth. From there the rockets are fired once again to ensure a safe re-rentry into Earth's atmosphere.

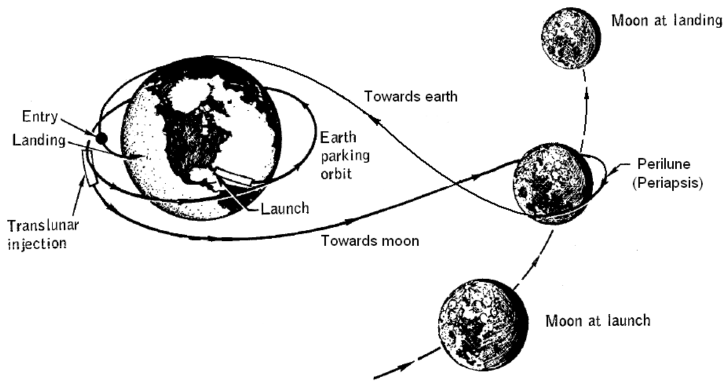

Figure 4: The lunar free return trajectory that the Apollo astronauts took to the Moon. Note that the Moon moves along its orbit around the Earth as the spacecraft approaches. The translunar injection burn must be timed just right so that the Moon is in the right place when the spacecraft arrives!

SOLAR SYSTEM EXPLORATION

To explore the Solar System, spacecraft first need to escape Earth's gravity. Once far enough from Earth, spacecraft can fire their rockets to enter a transfer orbit to another planet!

Earth Escape

To escape Earth's gravitational influence, the kinetic energy of a spacecraft must equal its gravitational potential energy. In this situation, the total energy of the spacecraft is zero. Recall that spacecraft traveling along bound orbits, such as circular or elliptical orbits, have negative energy. This is due to the fact that they are "trapped" in the Earth's gravitational potential well.

The velocity required to escape the gravitational influence of a massive body can be found by setting the specific energy to zero:

$$\begin{aligned} \mathcal{E} &= \frac{1}{2}v^2 - \frac{\mu}{r} \\ 0 &= \frac{1}{2}{v_{esc}}^2 - \frac{\mu}{r} \end{aligned} \tag{14}$$

The escape velocity can be expressed as:

$$v_{esc} = \sqrt{\frac{2\mu}{r}} \tag{15}$$

This means that a spacecraft orbiting a planet at a distance r will need a velocity equal to v_{esc} to escape the gravity of the planet.

*Orbit Insertion**

This section is optional!

Imagine a spacecraft is traveling along a Hohmann transfer orbit to Mars. It arrives at Mars' circular orbit around the Sun and is approaching the planet. What is going to happen? The spacecraft is actually traveling too fast to enter a closed orbit around Mars because its velocity exceeds Mars' escape velocity. As the spacecraft approaches the planet, it must turn around and fire its rockets against the direction of its motion in order to slow down. This is called an *orbit insertion burn*. Figure 5 shows a diagram of an orbit insertion burn for a spacecraft approaching Mars.

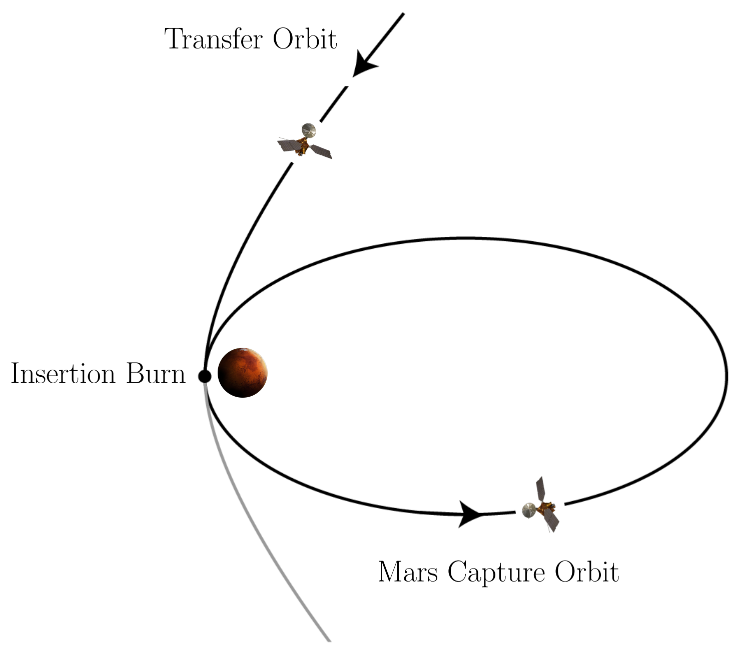

Figure 5: Mars orbit insertion. The spacecraft approaches Mars on its transfer orbit. At a point close to the planet, the spacecraft completes an insertion burn,. The burn slows the spacecraft down so that it can enter the Mars capture orbit, which is an ellipse.

*Fly-By Maneuvers**

This section is optional!

Spacecraft need a lot of "delta-v" to explore the outer Solar System. For example, a Hohmann transfer to Jupiter requires 14.4 km/s! This means that a spacecraft needs to carry a significant amount of propellant, which leaves less mass for useful scientific instruments. Is there a better way to get to the outer planets without using so much propellant?

Yes, there are better ways! One method uses "gravity fly-by" maneuvers. This is when a spacecraft closely approaches a planet, completes a partial orbit, and flies off into space towards another planet. It's sometimes called a "gravity sling-shot" because the spacecraft gains speed as a result of the maneuver. As the spacecraft approaches the planet, it accelerates because the planet exerts strong gravitational forces on the spacecraft. The spacecraft moves faster than the escape velocity of the planet, so it only completes a partial orbit before flying off into space. The partial orbit changes the direction of the spacecraft trajectory, so the spacecraft can leave the planet with higher velocity, heading in a direction towards its desired destination. Check out this **YouTube video** that shows a cool demo of gravity fly-bys!

Gravity fly-bys were used extensively for the Voyager missions in the 1970s. At that time, scientists realized that a special alignment of the planets was about to take place. With such an alignment, they would be able to use multiple gravity fly-bys to explore several of the outer planets. For example, Voyager 2 first visited Jupiter and used a fly-by to continue on to Saturn. It flew by Saturn, took lots of cool pictures, and continued on to Uranus. It flew by Uranus so that it could reach Neptune. Now Voyager 2 is exploring the outer reaches of the Solar System. Check out this **YouTube video** that shows Voyager 2's trajectory through the Solar System.

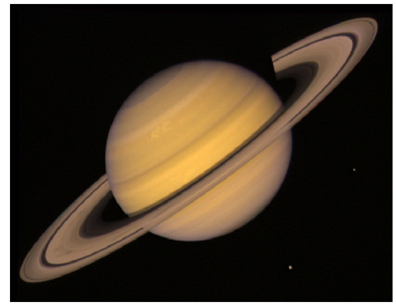

Figure 6: Saturn photo, captured by Voyager 2.

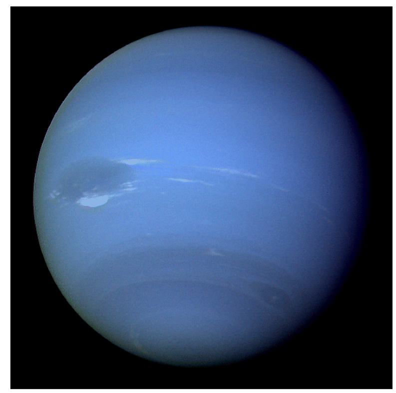

Figure 7: Neptune photo, captured by Voyager 2.