INTRODUCTION TO ROCKET PROPULSION

L5: Launch Trajectory Analysis

How do we predict the path of a launch vehicle?
In this lecture, we will learn how to predict the trajectory of a launch vehicle as it ascends through Earth's atmosphere! We'll start by discussing the change in velocity required for launch and why the choice of launch location matters. Then we will define the forces acting on the launch vehicle and write the equations of motion. We'll learn that these equations of motion can't be solved by hand, which is why we need computers! We'll learn how to rewrite the equations of motion using Euler's method so we can numerically integrate the trajectory from launch pad to orbit. In the lab, we'll write some programs to calculate launch trajectories!

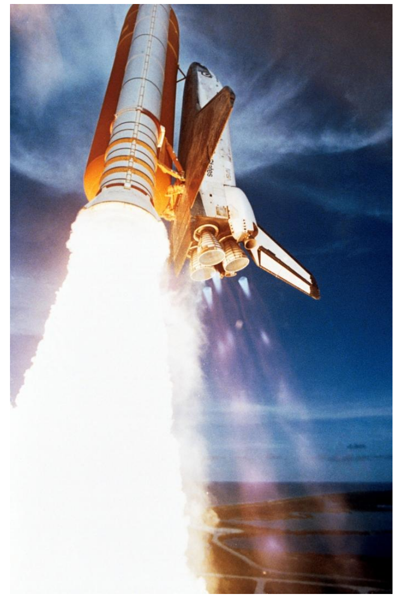

Figure 1: The Space Shuttle lifting off the launch pad.

LEARNING GOALS:

1. Explain why launch location is important and calculate the delta-v savings for different latitudes.
2. Mathematically describe the forces acting on a launch vehicle. Draw a diagram of a launch vehicle with the forces and coordinate system labeled.
3. Write the equations of motion for a launch vehicle.
4. Solve for the launch vehicle trajectory using Euler's method of numerical integration.

LAUNCH TO ORBIT

We spent the last several lectures learning about orbits and orbit transfers. In this lecture, we'll investigate how satellites and spacecraft are launched into orbit. We need a *launch vehicle* to send a satellite into orbit around the Earth. A launch vehicle is typically large, weighing approximately one million kilograms, and is propelled by chemical rockets. Approximately 90% of the mass of a launch vehicle is chemical propellant.

The function of a launch vehicle is to bring payload, such as a satellite, to the desired orbital altitude with the appropriate orbital velocity. Many satellites are delivered by launch vehicles to Low Earth Orbit (LEO), which ranges from 100 km to 1000 km above Earth's surface. For example, let's assume that a satellite needs to be delivered to a circular orbit with an altitude of 200 km. Using the equation for the velocity of a circular orbit, we can find that the velocity of the satellite needs to be 7.78 km/s. The launch vehicle must bring the satellite to a point 200 km in altitude, traveling parallel to Earth's surface at a speed of 7.78 km/s.

This is a challenging task! Launch vehicles must precisely deliver the correct change in velocity to the payload to place it in the correct orbit. Notice that the orbital velocity at a 200 km altitude is quite high - it's almost 8000 m/s! Remember that the larger the change in velocity, the more propellant is required to deliver that change in velocity. Launch vehicles use chemical rockets, which provide approximately 200 s to 450 s of specific impulse. If we assume $\Delta v = 7780$ m/s and $I_{sp} = 400$ s, then we can use the Ideal Rocket Equation to find that more than 86% of the launch vehicle mass must be propellant.

Note that the Δv used in this example is just a rough estimate. We need to account for the launch location, gravity losses, and drag losses in order to obtain an accurate estimate of the propellant mass required.

Launch Location

The launch location needs to be specified in order to determine the change in velocity the launch vehicle must deliver. Remember that on Earth's surface, we are moving relative to the center of the Earth! We don't notice this motion, but it's there and it can help give launch vehicles a boost into space. The Earth rotates from west-to-east along its axis, which connects the north and south poles. It takes 24 hours for the Earth to complete one revolution. With this information, we can calculate the rotational velocity at any point on Earth's surface.

Let's first consider the rotational velocity at the equator. Figure 2 shows a diagram of the Earth with the rotational velocity of the equator, $\vec{v}_{eq}$, labeled. The angular velocity of Earth's rotation along its axis is Ω, which is equal to 2π divided by the period of Earth's rotation. We can calculate the magnitude of the rotational velocity at the equator by dividing the distance traveled in one revolution by the time it takes to complete a revolution. The distance traveled is the circumference of the Earth at the equator, which is $2\pi R_E$. Therefore, the magnitude of the rotational velocity at the equator is:

$$\begin{aligned} v_{eq} &= \frac{2\pi R_E}{\mathbb{T}} \\ &= \frac{2\pi\,(6378\,\text{km})}{24\,\text{hours}} \\ &= 464\,\text{m/s} \end{aligned} \tag{1}$$

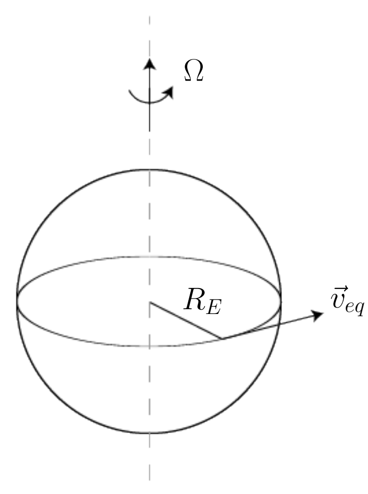

Figure 2: Rotational velocity at the equator. The velocity is perpendicular to the radial direction and is parallel to Earth's surface.

The surface at the equator rotates at 464 m/s relative to the center of the Earth. If we launch from the equator to the east, we can subtract 464 m/s from the delta-v required of the launch vehicle!

What happens at other latitudes? We can use a similar equation as before; we just need to calculate the distance traveled a bit differently. Figure 3 shows the rotational velocity at the equator and at a specified latitude, Λ. To compute the rotational velocity at any latitude, we need to determine the distance traveled during one revolution at that latitude. Notice in Figure 3 that the distance traveled during one revolution gets smaller as the latitude increases. If one were to stand at the north pole, for example, the distance traveled would be zero!

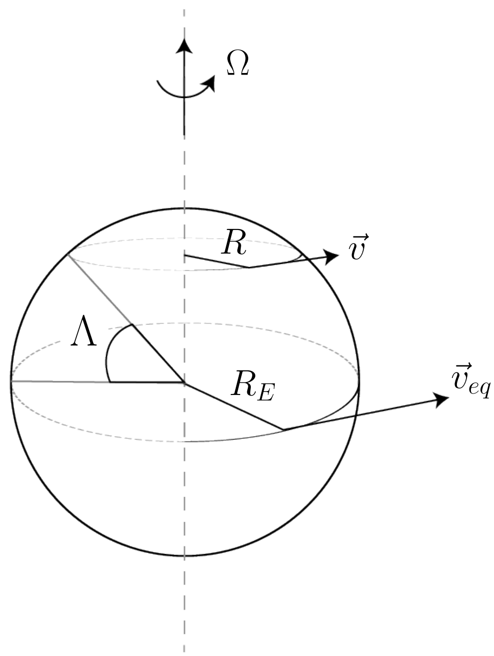

Figure 3: Rotational velocity at a specified latitude. The distance traveled in one revolution at higher latitudes is smaller than that at the equator. Therefore, the rotational velocity at higher latitudes is less than that at the equator.

The distance traveled during one revolution is given by $2\pi R$, where R is the distance between the Earth's surface and Earth's axis of rotation. At a latitude of Λ, the distance between the surface and the axis of rotation is $R_E \cos \Lambda$. Therefore, the rotational velocity, v_r, at a given latitude is:

$$\begin{aligned} v_r &= \frac{2\pi R}{\mathbb{T}} \\ &= \frac{2\pi R_E \cos \Lambda}{\mathbb{T}} \\ &= v_{eq} \cos \Lambda \end{aligned} \tag{2}$$

Example 1:

Let's consider Kennedy Space Center (KSC) in Cape Canaveral, Florida as the launch location. The latitude of KSC is 28.5°. The rotational velocity at KSC is:

$$\begin{aligned} v_r &= v_{eq} \cos \Lambda \\ &= (464\,\mathrm{m/s}) \cos 28.5^\circ \\ &= 408\,\mathrm{m/s} \end{aligned} \tag{1.1}$$

Therefore if we launch from KSC to the east, we can subtract 408 m/s from the required change in velocity from the launch vehicle. It may not seem like much but saving 408 m/s can create room for more payload!

Let's recalculate the propellant mass percentage from our earlier example. The change in velocity is now 7780 m/s minus 408 m/s, which is 7372 m/s. The launch vehicle must be 85% propellant by mass, which is a 1% savings. This may seem small, but it can translate to hundreds of kilograms of extra payload depending on the launch vehicle mass.

FORCES ON A LAUNCH VEHICLE

To predict the trajectory of a launch vehicle, we need to model the forces that act on the vehicle at all times. Figure 4 shows a diagram of a rocket and the forces acting on it.

The forces acting on the launch vehicle are as follows:

1. **Gravity.** The force of gravity pulls the vehicle towards the center of the Earth. It acts on a point on the launch vehicle called the *center of mass*. The magnitude of the gravitational force is equal to the weight of the vehicle, W:

$$W = mg \tag{3}$$

   where m is the vehicle mass, which is a function of time, and g is the acceleration due to gravity, which is a function of altitude.

2. **Thrust.** The thrust force, T, from the rocket engines acts along the axis of the launch vehicle, accelerating it forwards. The thrust can change slightly over time, if needed, and can be directed at different angles, usually within a few degrees of the axis of the launch vehicle.

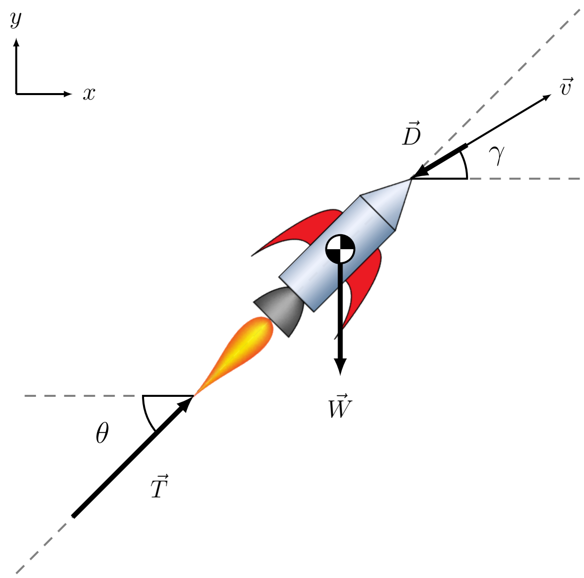

Figure 4: **Forces on a launch vehicle.** The thrust force, T, is aligned with the axis of the launch vehicle while the drag force, D, is not. Instead, the drag acts in opposition to the velocity of the vehicle. The velocity vector is at an angle γ from the horizontal, which is called the *flight path angle*. The angle between the axis of the vehicle and the horizontal is denoted by θ.

3. **Drag.** The atmosphere exerts a frictional force on the launch vehicle that acts opposite to the direction of motion, called drag. We can calculate the magnitude of the drag force, D, using the following expression:

$$D = \frac{1}{2}\rho v^2 A\, c_D \tag{4}$$

where ρ is the density of the atmosphere, v is the vehicle speed, A is the cross-sectional area of the vehicle, and c_D is the drag coefficient.

The cross-sectional area is the area of the largest slice through the vehicle that is perpendicular to the direction of motion of the vehicle. For example, the cross-sectional area of a baseball of radius r is πr^2, which is the area of the largest circular slice through the ball. Note that drag coefficients are determined experimentally using wind-tunnel tests. They vary depending on the vehicle geometry, speed, and surface roughness, among other things.

The drag force increases with increasing vehicle speed and atmospheric density. Fortunately, launch vehicles don't move very fast through the dense lower atmosphere. Launch vehicles reach their highest speeds at high altitudes where the atmosphere is nearly vacuum, so the drag force isn't very strong.

Now that we know the forces acting on the vehicle, we can determine the sum of the forces. We'll use the sum of the forces in the next section when we calculate the launch vehicle trajectory. We will sum the forces up in two directions:

1. **Parallel direction**. This direction is parallel to the flight path of the vehicle, or in other words, the same direction as the velocity.
2. **Perpendicular direction**. This is the direction perpendicular to the flight path of the vehicle, pointing approximately downward towards the surface of the Earth.

For our analysis, we will assume that $\theta = \gamma$. Note that this may not be true for vehicles that experience torques which rotate the axis of the vehicle relative to the velocity vector. The sum of the forces in the parallel direction is:

$$\sum F_{\parallel} = T - W \sin \gamma - D \tag{5}$$

The sum of the forces in the perpendicular direction is:

$$\sum F_{\perp} = -W \cos \gamma \tag{6}$$

TRAJECTORY ANALYSIS

Equations of Motion

We can use Newton's Second Law to solve for the trajectory of the launch vehicle. Remember that Newton's Second Law states that the sum of the forces acting on a body is equal to the mass times the acceleration of that body. We will write Newton's Second Law for the two directions of motion we considered in the last section.

The equation of motion in the parallel direction is:

$$\sum F_{\parallel} = m a_{\parallel} \tag{7}$$

where $a_{\parallel}$ is the acceleration in the parallel direction. The equation of motion in the perpendicular direction is:

$$\sum F_{\perp} = m a_{\perp} \tag{8}$$

where $a_{\perp}$ is the acceleration in the perpendicular direction.

Notice that $a_{\parallel}$ is parallel to the velocity vector, which means that it's the time rate of change of the speed:

$$a_{\parallel} = \frac{dv}{dt} \tag{9}$$

In contrast, $a_\perp$ is perpendicular to the velocity vector and thus the flight path of the vehicle. The perpendicular acceleration is a function of the vehicle speed, altitude, flight path angle, and rate of change of the flight path angle:

$$a_\perp = \frac{v^2}{R_E + h}\cos\gamma - v\frac{d\gamma}{dt} \tag{10}$$

where h is the altitude of the vehicle. This equation is used for motion along any curved path and is a generalization of the centripetal acceleration, which is used for circular motion.

When a body follows a circular path, the flight path angle is always zero since the position and velocity vectors are always perpendicular to each other. Therefore, the rate of change of the flight path angle is also zero. By plugging in $\gamma = 0$ and $d\gamma/dt = 0$ in equation 10, we recover the expression for the centripetal acceleration: $a_c = v^2/r$, where $r = R_E + h$.

Finally, using the sums of the forces from the previous section, we can write the two equations of motion as:

$$m\frac{dv}{dt} = T - \frac{1}{2}\rho v^2 A c_D - mg\sin\gamma \tag{11}$$

$$m\left(\frac{v^2}{R_E + h}\cos\gamma - v\frac{d\gamma}{dt}\right) = -mg\cos\gamma \tag{12}$$

The equations of motion are even more complex since the air density and the gravitational acceleration are both functions of the vehicle altitude. Also, the mass of the vehicle is a function of time.

We can express the gravitational acceleration, g, as a function of altitude, h, using the following expression:

$$g(h) = g_0\left(\frac{R_E}{R_E + h}\right)^2 \tag{13}$$

where g_0 is the acceleration due to gravity on Earth's surface.

The atmospheric density, ρ, as a function of altitude, h, can be modeled using the following equation:

$$\rho(h) = \frac{p_0}{T_0}\frac{\mathfrak{M}}{\mathfrak{R}}\left(1 - \frac{Lh}{T_0}\right)^{\frac{g_0\mathfrak{M}}{L\mathfrak{R}} - 1} \tag{14}$$

where p_0 is the atmospheric pressure at sea-level, T_0 is the atmospheric reference temperature at sea-level, $\mathfrak{M}$ is the mean molecular mass of air, $\mathfrak{R}$ is the ideal gas constant, and L is the temperature lapse rate, which is how quickly the atmospheric temperature decreases with altitude. This model is valid up to an altitude of 40 km.

Finally, we can express the mass of the vehicle as a function of time with the following equation:

$$m(t) = m_0 - \dot{m}t \tag{15}$$

where m_0 is the initial mass of the vehicle and $\dot{m}$ is the mass flow rate of the rockets. We'll assume that the mass flow rate is constant.

Another complication that we haven't yet considered is how to model the drag force. The coefficient of drag depends on the vehicle speed and generally cannot be described using an analytical expression. Instead, the coefficient of drag needs to be looked up from a table. So how are we supposed to integrate the equations of motion if one of the variables needs to be looked up from a table?! Even if the coefficient of drag was a constant value, we still wouldn't be able to solve the equations analytically because the variables $v(t)$, $\gamma(t)$, and $h(t)$ are interdependent. To solve these equations, we need to use a different strategy, called *numerical integration*.

Numerical Integration

Numerical integration is an approach to solving differential equations that can't be solved analytically. Instead of solving for an analytical equation that describes the velocity of the launch vehicle as a function of time, we can compute numerical values of the velocity at discrete instants in time, starting from liftoff and ending when the rockets finish burning. This gives us an array of time values and a corresponding array of velocity values.

Numerical integration works by starting at some known initial conditions and using that information to compute the conditions a split-second later. For example, if we know the numerical values of the vehicle position and velocity at the moment the launch begins, $t = 0$, we can compute the numerical values of the forces acting on the vehicle at that moment in time. We can divide the forces by the mass of the vehicle to obtain the instantaneous accelerations. If we assume that the vehicle accelerates for a short duration of time, Δt, then we can estimate the change in velocity of the vehicle. We can estimate the change in position of the vehicle by using the current value of the velocity, since the velocity is the time rate of change of the position.

We can add the change in velocity and change in position to the current values of the velocity and position, respectively. This yields the velocity and position at a moment in time later, specifically at $t = \Delta t$. We can repeat this process by using the new position and velocity to compute new values of the forces acting on the vehicle at $t = \Delta t$. Then, we can compute the change in velocity and change in position at $t = \Delta t$. From there, we can compute the position and velocity a moment later, which is $t = 2\Delta t$. We can do this again and again until we find that the vehicle reaches orbit.

This approach sounds tedious, and indeed it is. But it's the only way that we can determine the motion of a launch vehicle through Earth's atmosphere. The human computers at NASA, which we are reading about in the novel *Hidden Figures*, used this approach too! Today, we have the advantage of using laptops with considerable computing power. In the 1950s, electronic computing was in its infancy, so the human computers had to do all of the computations by hand. Later on, electronic computers became more advanced, relatively speaking, so the human computers were able to write programs to compute trajectories. They became the first computer scientists at NASA!

Figure 5: The human computers at NACA, later NASA, used Friden calculators. These bulky machines could do basic arithmetic, which was cutting edge at the time.

Before we numerically analyze launch trajectories, let's use numerical integration to solve a simple equation:

$$\frac{dy}{dt} = f(t) \tag{16}$$

We want to find $y(t)$. Normally we would solve the equation by integrating the right-hand side:

$$y(t) = \int f(t)\,dt \tag{17}$$

where $f(t)$ can be thought of as the *time rate of change* of y. Let's assume that we can't solve the integral analytically, so instead we need to use numerical integration.

Assume that we can calculate the numerical value of $f(t_i)$, where t_i is a specific value of t. We can use $f(t_i)$ to calculate the value of y at a later value of t. To do this, let's increase the value of t by a small amount, Δt, such that the value of t at the next timestep is $t_{i+1} = t_i + \Delta t$. Then, we can use Euler's Method, which is a linear approximation, to compute the change in y:

$$\Delta y \approx f(t_i)\,\Delta t \tag{18}$$

Now we can calculate the value of y at t_{i+1} by adding Δy to $y(t_i)$:

$$\begin{aligned} y(t_{i+1}) &= y(t_i) + \Delta y \\ &= y(t_i) + f(t_i)\,\Delta t \end{aligned} \tag{19}$$

As long as we know the value of y and f at time t_i, then we can calculate the value of y at the next timestep, which is $y(t_{i+1})$. This process can continue on because now we can calculate $f(t_{i+1})$ using $y(t_{i+1})$. Keeping Δt the same, we can calculate a new value for Δy and compute $y(t_{i+2})$:

$$\begin{aligned} y(t_{i+2}) &= y(t_{i+1}) + \Delta y \\ &= y(t_{i+1}) + f(t_{i+1})\,\Delta t \end{aligned} \tag{20}$$

We can repeat this process again and again as needed.

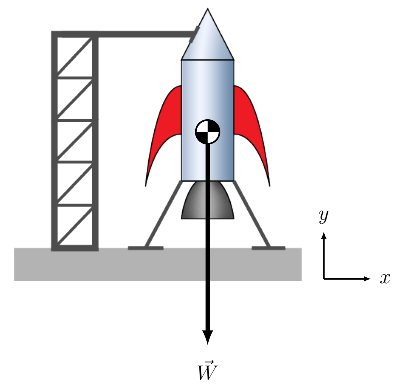

Figure 6: Before the engines ignite, the only force acting on the launch vehicle is gravity. When the engines fire, they produce a thrust force opposite in direction of the vehicle weight, $\vec{W}$. Notice that the thrust of the rockets must be greater than the initial weight for the vehicle to lift off!

Numerical Integration of Rocket Trajectories

There are four variables that we need to keep track of while integrating the trajectory of a rocket: the mass $m(t)$, the velocity $v(t)$, the flight path angle $\gamma(t)$, and the altitude $h(t)$. To use Euler's Method, we need to find the *time rates of change* of these four variables. This means we need to define four equations to integrate!

Let's find the time rate of change of the velocity by rewriting the equation of motion in the parallel direction:

$$\begin{aligned} \frac{dv}{dt} &= \frac{1}{m}\left(T - mg\sin\gamma - \frac{1}{2}\rho v^2 A c_D\right) \\ &= \frac{1}{m}\sum F_\parallel \\ &= f_\parallel(t) \end{aligned} \tag{21}$$

where $f_\parallel(t)$ is the sum of the forces in the parallel direction divided by the mass. Ultimately $f_\parallel$ is a function of time since all the variables it depends on also depend on time.

Now let's find the time rate of change of the flight path angle by rewriting the equation of motion in the perpendicular direction as:

$$\begin{aligned} \frac{d\gamma}{dt} &= \left(\frac{v}{R_E + h} - \frac{g}{v}\right)\cos\gamma \\ &= f_\perp(t) \end{aligned} \tag{22}$$

where $f_\perp(t)$ is the time rate of change of the flight path angle. We can also conclude that $f_\perp$ is a function of time since all the variables it depends on also depend on time.

If we assume the mass flow rate of the rockets is constant, the time rate of change of the vehicle mass is given by

$$\frac{dm}{dt} = -\dot{m} \tag{23}$$

Finally, we need to find the time rate of change of the vehicle altitude. We can relate the change in the distance traveled along the flight path, ds, to the change in altitude, dh, using the triangle in Figure 7. The time rate of change of the altitude is:

$$\begin{aligned} \frac{dh}{dt} &= \frac{ds}{dt}\sin\gamma \\ &= v\sin\gamma \end{aligned} \tag{24}$$

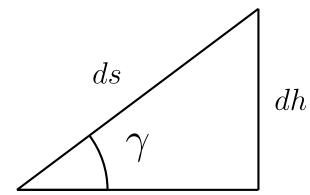

Figure 7: The triangle shows the relationship between the change in flight path distance, ds, flight path angle, γ, and the change in altitude, dh.

where the time rate of change of the flight path distance, ds/dt, is equal to the vehicle velocity. We found all four equations!

Now we can apply Euler's method to equations 21-24 to numerically integrate them over time. We'll start at $t = 0$, when the launch vehicle is on the launch pad. At $t = 0$, the speed of the vehicle is zero, the flight path angle is approximately 90°, the mass is m_0, and the altitude is zero. We know enough information to calculate the initial values of the four rates of change.

Given these initial conditions, we can increase the time by Δt. Using Euler's method, we can calculate the numerical value of the velocity, flight path angle, mass, and altitude at the next timestep. The following expressions show how Euler's Method can be applied to equations 21-24:

$$v(t_{i+1}) = v(t_i) + f_{\|}(t_i)\Delta t \tag{25}$$

$$\gamma(t_{i+1}) = \gamma(t_i) + f_{\perp}(t_i)\Delta t \tag{26}$$

$$m(t_{i+1}) = m(t_i) - \dot{m}\,\Delta t \tag{27}$$

$$h(t_{i+1}) = h(t_i) + v(t_i)\sin\gamma(t_i)\,\Delta t \tag{28}$$

Once we have values for v, γ, m, and h at the next timestep, we can calculate new values for the rates of change. Then we can apply Euler's Method again and again to obtain the numerical values of v, γ, m, and h at discrete instants in time. Solving for these quantities at each timestep is a laborious process, which is why we use computer programs. In the lab, we'll apply Euler's method to a simpler problem. Then, for your final project, you'll work on a program to compute the trajectory of a launch vehicle using equations 25-28.