INTRODUCTION TO ROCKET PROPULSION

L8: Chemical Rocket Nozzles

How do chemical rocket nozzles accelerate the exhaust?
In this lecture, we will investigate how rocket nozzles accelerate exhaust gases to high velocities. First we'll consider a classic problem: water flowing through a garden hose. Then we'll consider gas flowing through a rocket nozzle. We'll find that the exhaust gases are accelerated to supersonic speeds! Finally, we'll review some equations that will help us design rocket nozzles and calculate the exhaust velocity.

Figure 1: Operational liquid bi-propellant rocket engine on a test stand. The converging-diverging nozzle accelerates the hot combustion gases to supersonic speeds. We'll learn about why rocket nozzles are shaped this way later in the lecture!

Learning Goals:

1. Explain what the conservation of mass is and how it pertains to fluid flow through channels.
2. Explain why converging-diverging rocket nozzles are shaped the way they are. Draw a nozzle and label where the flow is subsonic, sonic, and supersonic.
3. Calculate the nozzle area ratio given a pressure ratio.
4. Calculate the exhaust velocity of a chemical rocket engine.

GARDEN HOSE PROBLEM

We know from earlier lectures that fuel and oxidizer are mixed together in the combustion chamber of a bi-propellant chemical rocket engine. The mixture combusts, producing high temperature, high pressure gas. At this point, the combustion gases move slowly along the axis of the rocket engine towards the exit. To increase the thrust and specific impulse, we need to accelerate the gas to high velocities using a nozzle.

Let's start at the beginning. After the reaction is completed, the gaseous combustion products have low kinetic energy. For the purposes of our analysis, we can assume that the initial velocity of the gas is zero. However, the gas is at high temperature and pressure, which means that the gas has high potential energy. Remember that potential energy is essentially the ability to do work. To accelerate the gas, we need to convert the potential energy to kinetic energy. In a rocket engine, a nozzle is used to accomplish this task.

Before we learn about rocket nozzles, let's start by thinking about garden hoses. Have you ever played with a garden hose outside? When you open the faucet, the water gushes out the end of the hose. If you hold the hose horizontally, the water travels a short distance, maybe a few feet, before hitting the ground. It probably doesn't travel far enough to spray your friend who is standing 10 feet away.

Figure 2: Water flowing slowly out of a garden hose.

So what do you do to spray your friend? You probably cover part of the hose opening with your thumb. Then, all of the sudden, the water comes out of the hose at a much faster speed! A faster stream of water travels a much farther distance before hitting the ground - so you successfully spray your friend with water!

Figure 3: Water flowing quickly out of a garden hose. You can use your thumb to cover part of the exit area of a garden hose, which forces the water out at high velocity.

Conservation of Mass

Why does covering part of the end of a garden hose speed up the water spray? Let's introduce a key concept: *The Conservation of Mass*.

> DEFINITION 8.1 **The Conservation of Mass** states that mass is neither created nor destroyed within a closed system.

Let's consider how the conservation of mass pertains to the garden hose problem. We know that when we open the faucet, water flows into the hose at a constant rate. This means that the mass of water entering the hose per unit time is constant. We will call the mass flow rate of water going into the hose $\dot{m}_{in}$. We also know that water comes out the end of the hose. Let's call the mass flow rate of water exiting the hose $\dot{m}_{out}$.

Assuming that the hose isn't leaky, we can say that no water leaves the hose as it travels to the hose exit. We'll also assume that no water is added anywhere along the length of the hose. Therefore, the mass flow rate of water exiting the hose must equal the mass flow rate of water entering the hose. We can state the conservation of mass as:

$$\dot{m}_{in} = \dot{m}_{out} \tag{1}$$

This is a critical step in understanding why changing the area of the hose exit changes the speed of the water.

We need to find the relationship between the mass flow rate, the water speed, and the cross-sectional area of the hose. Consider the diagram in Figure 4, which shows an infinitesimally thin slice of the garden hose. Let's calculate the mass of water in that slice of hose, Δm. Assuming that the cross-sectional area, A, is constant, we can say that the mass is the density of the water times the volume of the slice of hose:

$$\begin{aligned}\Delta m &= \rho V \\ &= \rho\, \Delta x\, A\end{aligned} \tag{2}$$

where ρ is the density of the water and V is the volume of the slice. The volume of the slice is the length of the slice, Δx, times the cross-sectional area of the hose, A.

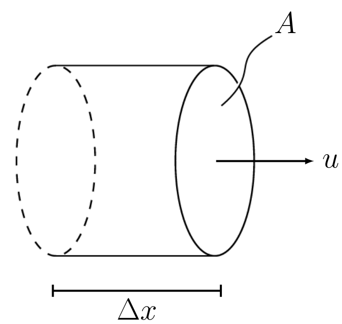

Figure 4: Slice of a garden hose. The length of the hose is Δx, the cross-sectional area of the hose is A, and the velocity of the water is u.

What is the mass flow rate of water through this section of hose? We can divide equation 2 by a small increment of time, Δt:

$$\frac{\Delta m}{\Delta t} = \rho A \frac{\Delta x}{\Delta t} \tag{3}$$

Right away we can recognize $\Delta m / \Delta t$ as the mass flow rate, $\dot{m}$. What about $\Delta x / \Delta t$? This is the velocity of the water, u, as it flows through the slice of hose. We can write equation 3 as:

$$\dot{m} = \rho u A \tag{4}$$

We have found that the mass flow rate of fluid through a channel is given by the product of the fluid density, the fluid velocity, and the area of the channel.

Now we can apply equation 4 to our garden hose problem. We can express the conservation of mass as:

$$\begin{aligned}\dot{m}_{in} &= \dot{m}_{out} \\ \rho_{in}\, u_{in}\, A_{in} &= \rho_{out}\, u_{out}\, A_{out}\end{aligned} \tag{5}$$

Let's assume that the density of the water remains constant, which means that $\rho_{in} = \rho_{out}$. Since the density cancels out, we can write equation 5 as:

$$u_{in} A_{in} = u_{out} A_{out} \tag{6}$$

The velocity of the water times the cross-sectional area of the hose must remain constant throughout the length of the hose.

Let's consider what happens when you cover the exit of the hose. Assume that your thumb covers half of the hose exit area. This means that $A_{out} = \frac{1}{2} A_{in}$. We can substitute this relationship into equation 6 and solve for the velocity:

$$u_{out} = 2 u_{in} \tag{7}$$

By covering half of the hose exit, the water is forced out the hose at twice the speed! In general, as the hose area decreases, the water velocity increases.

CONVERGING-DIVERGING ROCKET NOZZLES

Compressible Flow

What does this have to do with rocket engines? Well, the same principle works for a gas. As the flow area decreases, the speed of the gas increases. This is why the diameter of the nozzle gets smaller downstream of the combustion chamber. As the nozzle gets narrower, the gas speed increases significantly. However, we can't make the flow area infinitely small! At a certain point, making the nozzle smaller actually *slows down* the gas. This is the opposite of what we would expect! How can this be?

We made a critical assumption in the garden hose problem. We assumed that the density of the fluid is constant. However, this is not a good assumption for gases, which can be compressed or expanded. When a gas is compressed, its temperature, pressure, and density tend to increase. When a gas is expanded, its temperature, pressure, and density tend to decrease. For a gas in a rocket engine, we can only use equation 4, not equation 6.

As the gas flows downstream, the decreasing channel area accelerates the gas and compresses it. In other words, the gas travels at a faster speed and the density of the gas increases. Something very interesting happens to the gas as it gets denser and faster. The speed of the gas reaches a critical threshold: the speed of sound of the gas.

Quantifying how the speed and density of the gas change as a function of the channel area requires advanced analysis that we won't cover in this course. If you're curious about this, check out *Mechanics and Thermodynamics of Propulsion* by Philip G. Hill and Carl Peterson.

DEFINITION 8.2 The **speed of sound** is the speed at which pressure disturbances travel through a medium. The speed of sound in an ideal gas is given by the following equation:

$$a = \sqrt{\gamma R T} \tag{8}$$

where a is the speed of sound, γ is the specific heat ratio of the gas, R is the gas constant, and T is the temperature of the gas. Note that R is the universal gas constant divided by the molecular mass of the gas.

DEFINITION 8.3 The ratio of the speed of a moving body to the local speed of sound is called the **Mach number**. The Mach number is denoted by M and can be computed using the following equation:

$$\mathrm{M} = \frac{v}{a} \tag{9}$$

where v is the speed of the moving body and a is the local speed of sound. We use the term *local* to specify the speed of sound at the location of the moving body. In some situations, like in a rocket engine, the speed of sound changes depending on the location.

When the gas speed exceeds its own speed of sound, we say that the gas is *supersonic*. In this case the Mach number is greater than 1. When the gas speed is less than its own speed of sound, the gas is *subsonic* and has a Mach number between 0 and 1. When the gas is traveling exactly at its own speed of sound, the gas is *sonic* or *transonic* and has a Mach number of 1.

Figure 5: Me pretending like supersonic fluid flow makes total sense.

Gases behave very strangely when they are supersonic. For supersonic gases, when the channel area decreases, the speed of the gas decreases. When the channel area increases, the speed of the gas increases. This is the exact opposite of what happens for subsonic gases, which is a very non-intuitive result!

Rocket Nozzle Geometry

A chemical rocket engine, meaning the combustion chamber and nozzle, can be divided into three regions, which are labeled in Figure 6. The shape of the rocket nozzle is called *converging-diverging* because the flow area decreases in the first region, reaches a minimum at the second region, and increases throughout the third region.

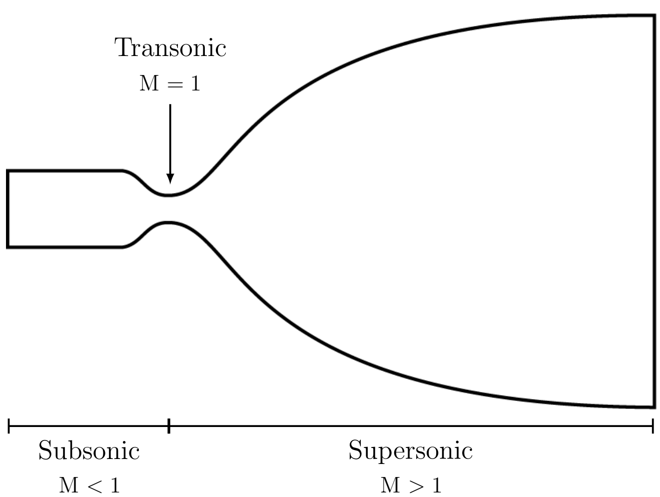

Figure 6: Converging-diverging chemical rocket nozzle with gas speed regions labeled.

After the combustion reaction completes, the gaseous products move slowly towards the exit of the rocket engine. To accelerate the gas, the flow area is decreased gradually until the minimum flow area is reached. During this initial acceleration phase, the gas is *subsonic*. The point where the flow area is minimum is called the *throat*. The flow area at the throat is chosen such that the gas velocity at the throat is equal to the local speed of sound. At this point the gas is *transonic*.

Finally, to accelerate the gas further, the area of the nozzle is increased after the throat. The increasing area accelerates the *supersonic* gas to even faster speeds. The exit area of the nozzle is made as large as possible in order to accelerate the gas to the highest possible exit velocity in order to maximize the thrust and specific impulse of the rocket engine.

Rocket Nozzle Equations

In this section, we will review some equations that will help us design a converging-diverging rocket nozzle. Don't worry about where the equations come from or about memorizing them. Just be prepared to use the equations in your calculations in the lab.

Throat Area

The area of the throat, A_t, is given by following equation:

$$A_t = \dot{m}\frac{\sqrt{RT_c}}{p_c\sqrt{\gamma}}\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{10}$$

where $\dot{m}$ is the mass flow rate of propellant entering the combustion chamber, T_c is the chamber temperature, p_c is the chamber pressure, γ is the specific heat ratio of the gaseous combustion products, and R is the gas constant.

The throat needs to be exactly the correct size to ensure that the exhaust gases travel at the speed of sound when they pass through the throat. Notice that the throat area only depends on the conditions in the combustion chamber. In deriving equation 10, we start with the initial conditions of the combustion gases, which are given by the conditions in the combustion chamber. Then, using the conservation of mass, momentum, and energy, we can determine how the speed of the gas increases as the channel area decreases. We can find where the gas velocity is equal to the local speed of sound and then solve for the channel area at that point, which is the desired throat area. Note: the speed of sound changes as the gas approaches the throat because the temperature of the gas decreases as the gas accelerates!

Area Ratio

The *area ratio* or *expansion ratio*, ϵ, of a chemical rocket nozzle is:

$$\epsilon = \frac{A_e}{A_t} \tag{11}$$

where A_e is the exit area and A_t is the throat area. The expansion ratio for a typical rocket engine is about 40. Note that the exit area of a nozzle can't be made arbitrarily large because the larger the exit area, the more massive the nozzle. At some point the performance gains from making the exit area larger are canceled out by the increased mass of the nozzle.

Given the conditions in the combustion chamber, we can use the conservation of mass, momentum, and energy to solve for the gas velocity, pressure, and temperature at the exit of the nozzle. This analysis allows to relate the area ratio to the ratio of the exhaust pressure to the chamber pressure, which is called the *pressure ratio*. The area ratio is related to the pressure ratio by the following equation:

$$\frac{A_e}{A_t} = \sqrt{\frac{\gamma-1}{\gamma+1}} \left(\frac{\gamma+1}{2}\right)^{\frac{1}{1-\gamma}} \left(\frac{p_e}{p_c}\right)^{-\frac{1}{\gamma}} \left(1 - \left(\frac{p_e}{p_c}\right)^{\frac{\gamma-1}{\gamma}}\right)^{-1/2} \tag{12}$$

where p_e is the exhaust pressure, p_c is the chamber pressure, and γ is the ratio of specific heats. Figure 7 shows a plot of the pressure ratio versus the area ratio as specified by equation 12. Notice that as the area ratio increases, the pressure ratio decreases. A larger area ratio nozzle is able to expand the exhaust gases more, which means the pressure drop through the nozzle is larger. Given the chamber pressure, we can use the area ratio to calculate the exhaust pressure of the gas. We need to know the exhaust pressure in order to calculate the thrust, specific impulse, and exhaust velocity.

Designing a rocket nozzle is an iterative process. We can start by using the minimum allowable exhaust pressure, which is specified by the flow separation condition:

$$p_e \Big|_{min} = 0.4 \, p_a \Big|_{max} \tag{13}$$

where p_a is the ambient pressure. The maximum allowable area ratio can be computed using the minimum allowable exhaust pressure. If the maximum area ratio is too large, perhaps because the mass of the nozzle would be too high, then a smaller area ratio can be chosen. The exhaust pressure for the chosen area ratio can be solved for using equation 13 using numerical methods.

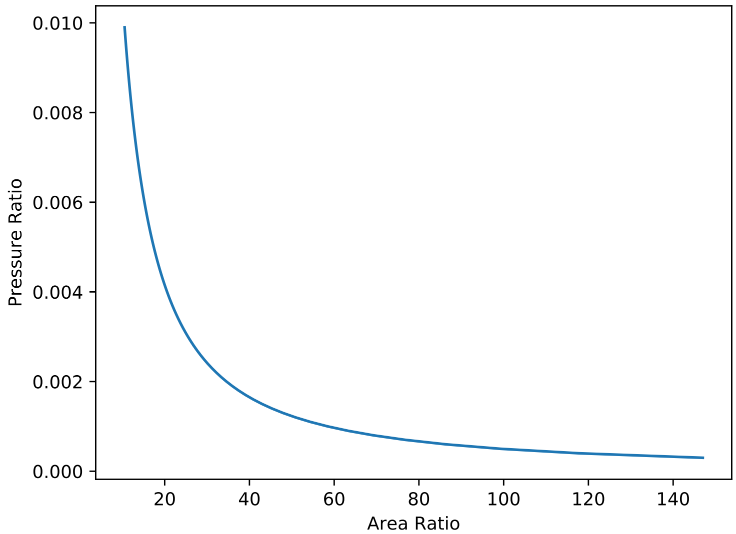

Figure 7: This graph shows the relationship between the area ratio and the pressure ratio as specified by equation 12. Note that a specific heat ratio of 1.26 was used.

Exhaust Velocity

If you're interesting in seeing how this equation is derived, read the next section, which is optional.

The exhaust velocity of a chemical rocket is given by the following expression:

$$u_e = \sqrt{\frac{2\gamma}{\gamma - 1}\frac{\mathfrak{R}}{\mathfrak{M}}T_c\left(1 - \left(\frac{p_e}{p_c}\right)^{\frac{\gamma-1}{\gamma}}\right)} \tag{14}$$

where γ is the specific heat ratio of the exhaust gas, $\mathfrak{R}$ is the universal gas constant, $\mathfrak{M}$ is the mean molecular mass of the exhaust gas, T_c is the chamber temperature, p_e is the pressure of the exhaust at the nozzle exit plane, and p_c is the chamber pressure.

Many of the quantities in equation 15 are set by the initial conditions of the combustion reaction. The chamber pressure is typically selected to be as high as possible. This ensures that the chemical combustion reaction is completed quickly with no leftover reactants. High chamber pressures also help increase the thrust and specific impulse of the rocket. However, the pressure cannot be arbitrarily high because the combustion chamber would need thick walls to withstand the high pressure. At a certain point the extra mass of the combustion chamber becomes a disadvantage.

The chamber temperature, or combustion temperature, can be determined given the fuel-to-oxidizer ratio and the chamber pressure. Typically, the chamber temperature is chosen to be as high as possible, since the higher the temperature, the higher the exhaust velocity.

However, the chamber temperature cannot be arbitrarily high, because the walls of the combustion chamber could melt!

The pressure ratio, p_e/p_c, is determined by the area ratio of the nozzle. Figure 7 shows that increasing the area ratio decreases the pressure ratio. From equation 14, we can see that the exhaust velocity increases as the pressure ratio decreases. To maximize the exhaust velocity, and thus the thrust and specific impulse, the area ratio of the nozzle is chosen to be as large as possible.

The specific heat ratio and the mean molecular mass of the exhaust gas are properties that can be determined from the composition of the exhaust gases. The exhaust gases are a combination of many types of molecular gases (such as carbon dioxide, carbon monoxide, water vapor, diatomic hydrogen, etc.) which are a product of the combustion reaction. The ratios of the reaction products are determined by the fuel-to-oxidizer ratio and the chamber pressure.

DERIVATION OF THE EXHAUST VELOCITY*

This section is optional!

The exhaust velocity of a chemical rocket can be found using the conservation of energy. We will assume that the increase in kinetic energy, KE, of the gas is equal to the decrease in its enthalpy, H:

$$\Delta \mathrm{KE} = -\Delta \mathrm{H} \tag{15}$$

To use equation 15, we must assume that no work is done on the gas by the surroundings and no work is done by the gas on the surroundings. We also must assume that no heat is transferred between the gas and its surroundings. The second assumption is not quite true because the hot gases transfer heat to the walls of the nozzle. However, the amount of heat lost is only $\sim$1% of the enthalpy of the gas. The other 99% of the enthalpy is converted into the kinetic energy of the exhaust gases, so our assumption is okay.

When it comes to analyzing gases, it's easier to work with quantities that are per unit mass. The kinetic energy per unit mass, ke is equal to $\frac{1}{2}u^2$, where u is the gas velocity. The change in enthalpy per unit mass is denoted by Δh. We can express the conservation of energy per unit mass as:

$$\Delta \mathrm{ke} = -\Delta h \tag{16}$$

DEFINITION 8.4 The **enthalpy** of a gas is essentially a measure of the ability of the gas to do work. It can be thought of as a potential energy. The enthalpy per unit mass can be mathematically expressed as:

$$h = e + pv \tag{17}$$

where e is the internal energy, p is the pressure, and v is the specific volume. Note: the specific volume is the inverse of the density of the gas: $v = 1/\rho$. Using the first law of thermodynamics, which is a statement of the conservation of energy, we can express the change in enthalpy, dh, as:

$$dh = de + pdv = c_p dT \tag{18}$$

where de is the change in internal energy, dv is the change in specific volume, c_p is the specific heat at constant pressure, and dT is the change in temperature.

Let's assume that the initial state of the gas is given by the conditions in the combustion chamber. The velocity of the gas in the combustion chamber is denoted by u_c. The final state of the gas is given by the conditions at the exit of the nozzle. Here, the velocity of the gas is the exhaust velocity, u_e. Therefore, the change in kinetic energy per unit mass of the gas is:

$$\Delta \text{ke} = \frac{1}{2}{u_e}^2 - \frac{1}{2}{u_c}^2 \tag{19}$$

We will assume that the velocity of the gas in the combustion chamber is approximately zero. Therefore the change in kinetic energy per unit mass is $\frac{1}{2}{u_e}^2$.

Now let's find the change in enthalpy per unit mass, Δh. The change in enthalpy per unit mass can be expressed in terms of the change in temperature of the gas:

$$\Delta h = c_p(T_e - T_c) \tag{20}$$

where c_p is the specific heat capacity at constant pressure, T_e is the temperature of the exhaust, and T_c is the chamber temperature.

Now let's put together equations 16, 19, and 20:

$$\begin{aligned} \Delta \text{ke} &= \Delta e \\ \frac{1}{2}{u_e}^2 &= -c_p(T_e - T_c) \end{aligned} \tag{21}$$

We can solve for the exhaust velocity:

$$u_e = \sqrt{2c_p\left(T_c - T_e\right)} \tag{22}$$

Next, we need to do some mathematical manipulation to write the equation in terms of the *temperature ratio* T_e/T_c:

$$u_e = \sqrt{2c_p T_c\left(1 - \frac{T_e}{T_c}\right)} \tag{23}$$

When analyzing a chemical rocket engine, the chamber temperature is usually known. The specific heat capacity, c_p, is also usually known. Typically c_p is written in terms of the ratio of specific heats, γ, the universal gas constant, $\mathfrak{R}$, and the molecular mass of the gas, $\mathfrak{M}$:

$$c_p = \frac{\gamma}{\gamma - 1}\frac{\mathfrak{R}}{\mathfrak{M}} \tag{24}$$

The temperature of the exhaust isn't used very often. Instead, the exhaust pressure is preferred because it's related to the geometry of the nozzle through the area ratio. We can relate the temperature ratio to the pressure ratio using the *isentropic relations*.

DEFINITION 8.5 The **isentropic relations** are a set of equations that describe how the temperature, pressure, and density of a compressible gas are related. We assume that the gas is an ideal gas, the entropy of the gas is constant, and no heat is absorbed or lost by the gas. The isentropic relations are as follows:

$$\frac{\rho_1}{\rho_2} = \left(\frac{T_1}{T_2}\right)^{1/\gamma - 1} = \left(\frac{p_1}{p_2}\right)^{1/\gamma} \tag{25}$$

where ρ is the density, T is the temperature, and p is the pressure. The quantities with subscript "1" denote the state of the gas at a particular location. The quantities with subscript "2" denote the state of the gas at a different location. These equations are essential for calculating the state of the gas as it expands through the rocket nozzle.

Using the isentropic relations, we can relate the temperature and pressure in the combustion chamber to the temperature and pressure at the nozzle exit:

$$\frac{T_e}{T_c} = \left(\frac{p_e}{p_c}\right)^{(\gamma-1)/\gamma} \tag{26}$$

Finally, using equations 24 and 26, we can write the exhaust velocity of a chemical rocket as:

$$u_e = \sqrt{\frac{2\gamma}{\gamma - 1}\frac{\mathfrak{R}}{\mathfrak{M}}T_c\left(1 - \left(\frac{p_e}{p_c}\right)^{\frac{\gamma-1}{\gamma}}\right)} \quad (27)$$

where γ is the specific heat ratio of the exhaust gas, $\mathfrak{R}$ is the universal gas constant, $\mathfrak{M}$ is the mean molecular mass of the exhaust gas, T_c is the chamber temperature, p_e is the pressure of the exhaust at the nozzle exit plane, and p_c is the chamber pressure.

Figure 8: We did it!